An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 1
Ordinary generating functions
(3)

IMSc

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From the previous lecture

12 January 2016

Path (or walk)

$$\omega = (s_0, s_1, \dots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

length n

(s_i, s_{i+1}) elementary step

ω is going from s_0 to s_n

notation: $s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation (weight)

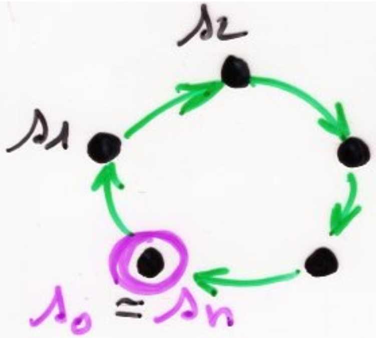
$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

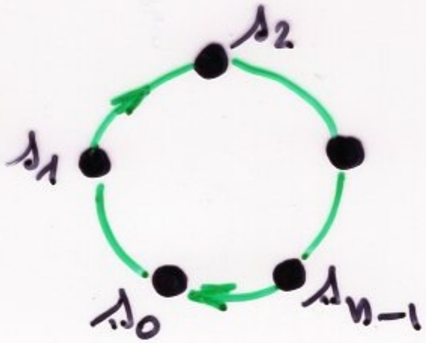
$$v : S \times S \to \mathbb{K}[x]$$

self-avoiding path (or walk)

all vertices $s_0, s_1, \dots, s_n$ are disjoint

elementary circuit $\omega = (s_0, \dots, s_n)$ with $s_0 = s_n$, all vertices are disjoint except $s_0 = s_n$.



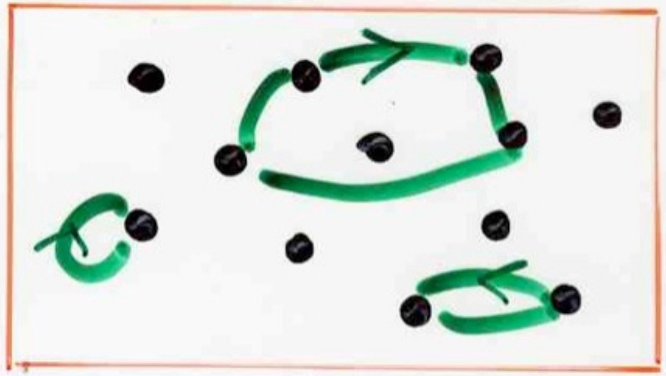


Cycle = elementary circuit up to a circular permutation of the vertices

Proposition

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} V(\omega) = \frac{N_{i,j}}{D}$$

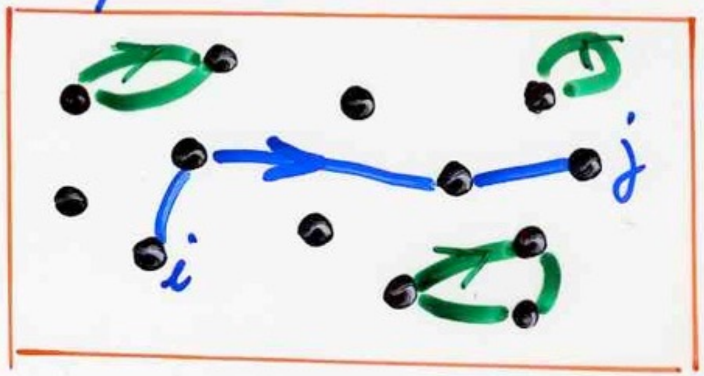
$$D = \sum_{\substack{\{\gamma_1,\dots,\gamma_r\} \\ \text{2 by 2 disjoint} \\ \text{cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta;\gamma_1,\ \ ,\gamma_r\}} (-1)^r \, v(\eta) \, v(\gamma_1) \cdots v(\gamma_r)$$

η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1,\dots,\gamma_r\}$ 2 by 2 disjoint cycles, and disjoint from η



linear algebra

proof

Lemma $S = \{1, 2, \dots, n\}$

$A = (a_{i,j})$ $n \times n$ matrix

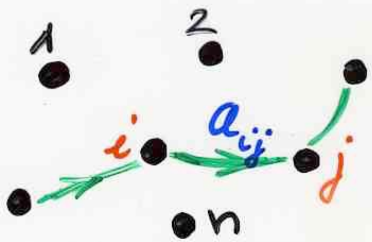
$$(I - A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

$$(I_n - A)^{-1}_{ij} = \frac{\mathrm{cof}_{ji}(I_n - A)}{\det(I_n - A)}$$

$I_n + A + A^2 + \dots + A^n + \dots$

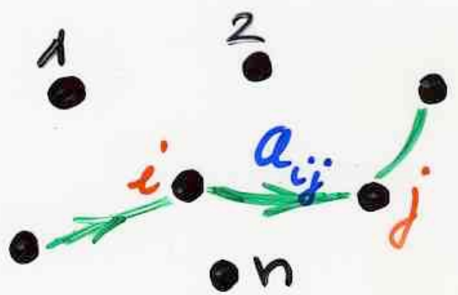
$A = (a_{ij})$



$$(I_n - A)^{-1}_{ij} = \frac{cof_{ji}(I_n - A)}{det(I_n - A)}$$

$$I_n + A + A^2 + \dots + A^n + \dots$$

$$A = (a_{ij})$$



bijective proof of the identity

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

bijective proof of

$$\left(\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega)\right) D = N_{ij}$$

<u>notations</u>.

- $Ch(i,j)$ set of paths on S going from i to j
- Det set of configurations $\{\gamma_1, \dots, \gamma_r\}$ of cycles on S, 2 by 2 disjoints
- $Cof(i,j) \subseteq Ch(i,j) \times Det$ set of configurations $(\eta ; \{\gamma_1, \dots, \gamma_r\})$ with η self-avoiding walk going from i to j, disjoint from $\gamma_1, \dots, \gamma_r$ and $\{\gamma_1, \dots, \gamma_r\} \in Det$
- $E(i,j) = \left(Ch(i,j) \times Det\right) \setminus Cof(i,j)$

Proposition There exist an involution φ

$$\varphi : E(i,j) \longrightarrow E(i,j)$$

such that, if $\varphi(\omega;\{\gamma_1,\dots,\gamma_r\}) = (\omega';\{\gamma'_1,\dots,\gamma'_s\})$, then

(i) $v(\omega)\, v(\gamma_1) \cdots v(\gamma_r) = v(\omega')\, v(\gamma'_1) \cdots v(\gamma'_s)$

(ii) $s = r \pm 1$

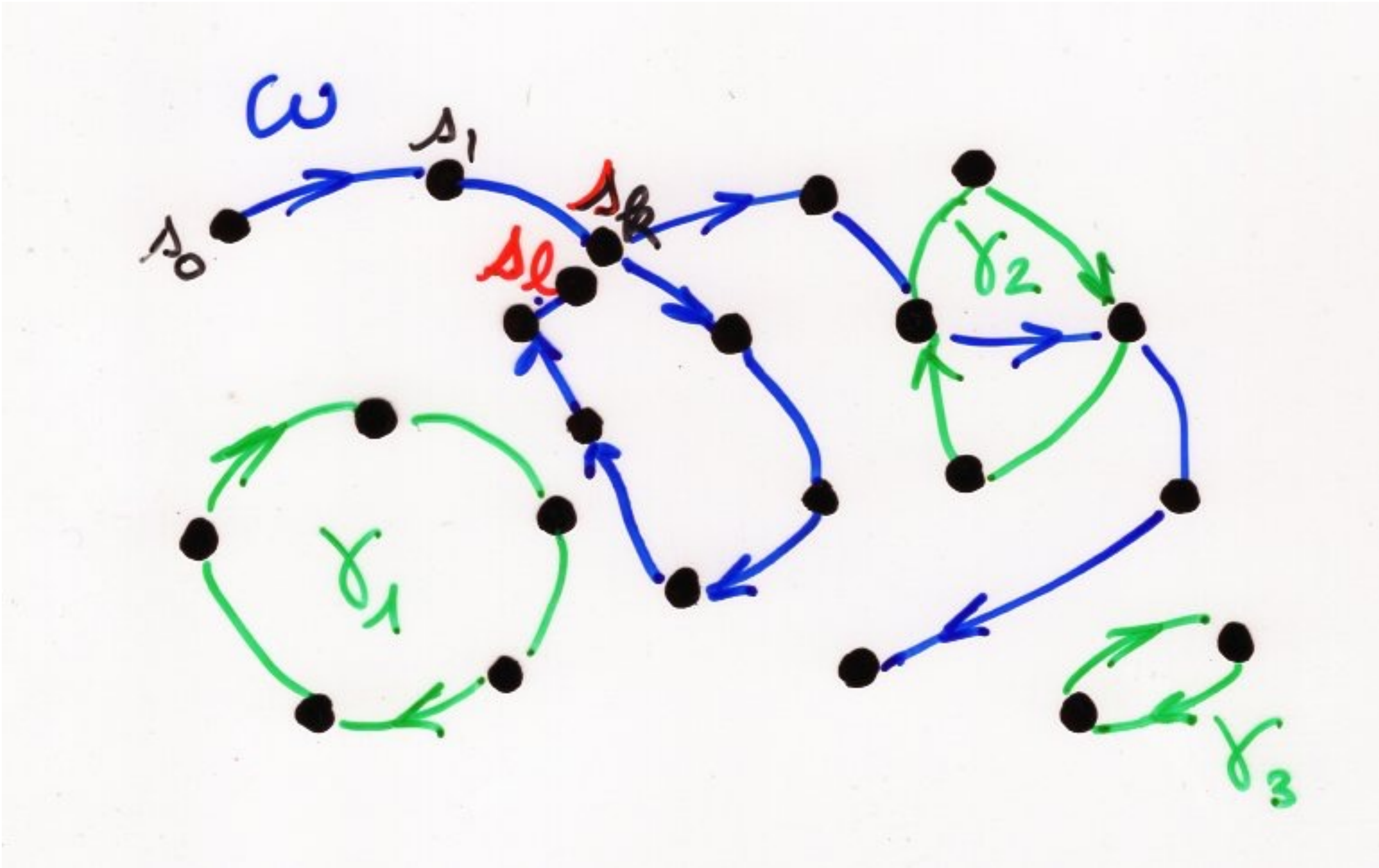
<u>Construction</u> of the involution φ

Let $\xi = (\omega ; \{\gamma_1, \dots, \gamma_r\}) \in E(i,j)$

with $\omega = (s_0, \dots, s_n)$

Let ℓ be the smallest integer $0 \leq \ell \leq n$ such that:

or $\begin{cases} \text{(i) } \exists k,\ 0 \leq k < \ell \text{ with } s_k = s_\ell \\ \text{(ii) } s_\ell \text{ belongs to one of the cycles } \gamma_1, \dots, \gamma_r \end{cases}$

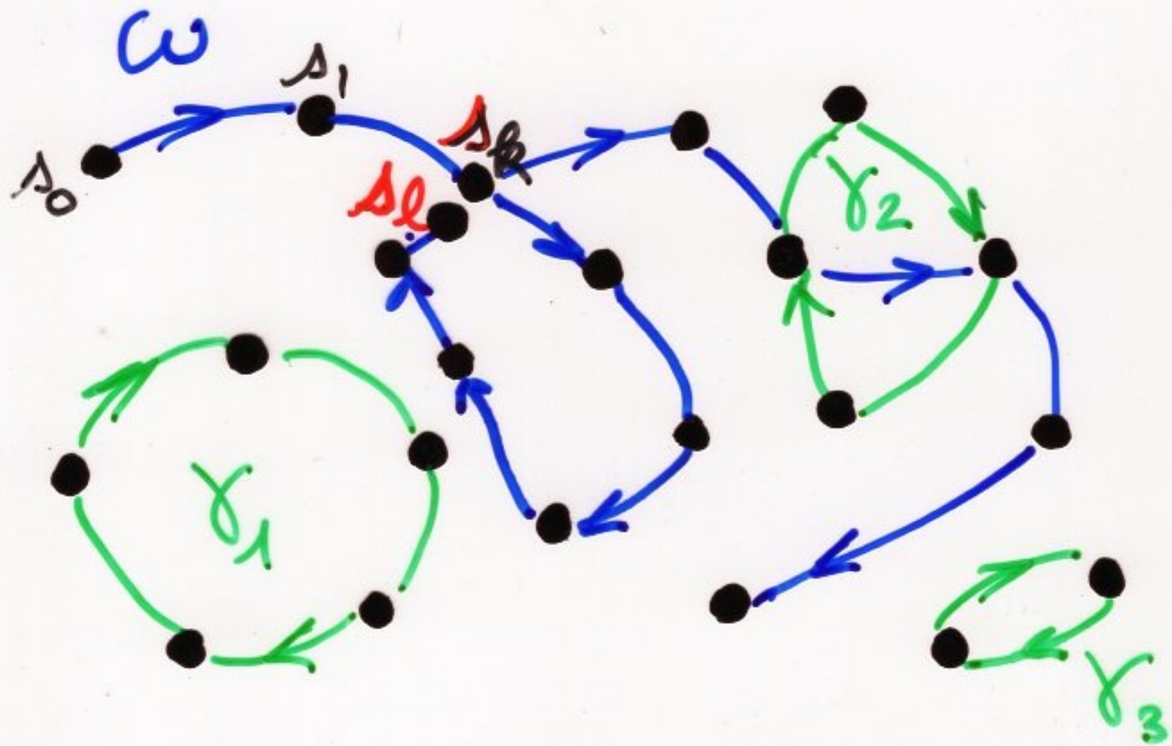


ℓ exists $\Leftrightarrow$ $\xi \notin \mathrm{Cof}(i,j)$

ℓ cannot satisfies both (i) and (ii) case (i) $\varphi(\xi) = (\omega' ; \{\gamma_1, \ldots, \gamma_r, \gamma\})$

with $\omega' = (s_0, \ldots, s_{k-1}, s_\ell, \ldots, s_n)$

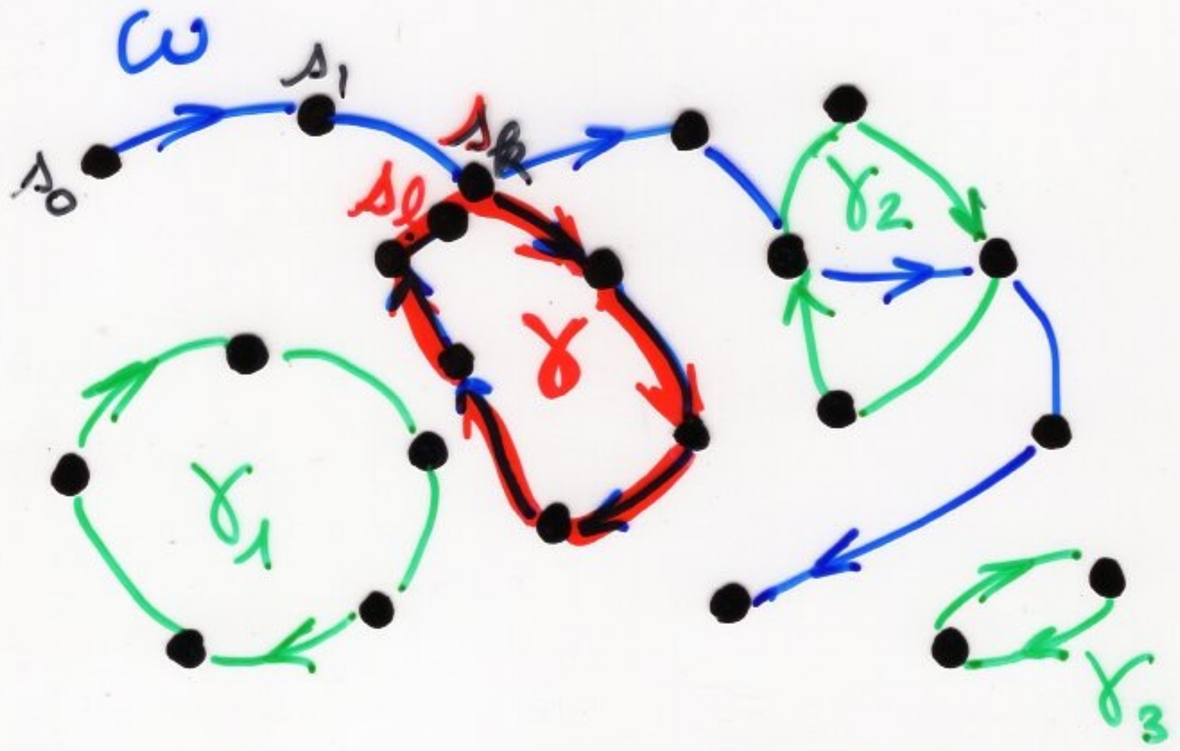
$\gamma = (s_k, s_{k+1}, \ldots, s_{\ell-1})$



case (i) $\varphi(\xi) = (\omega'; \{\gamma_1, .., \gamma_r, \gamma\})$

with $\omega' = (s_0, .., s_{k-1}, s_\ell, .., s_n)$

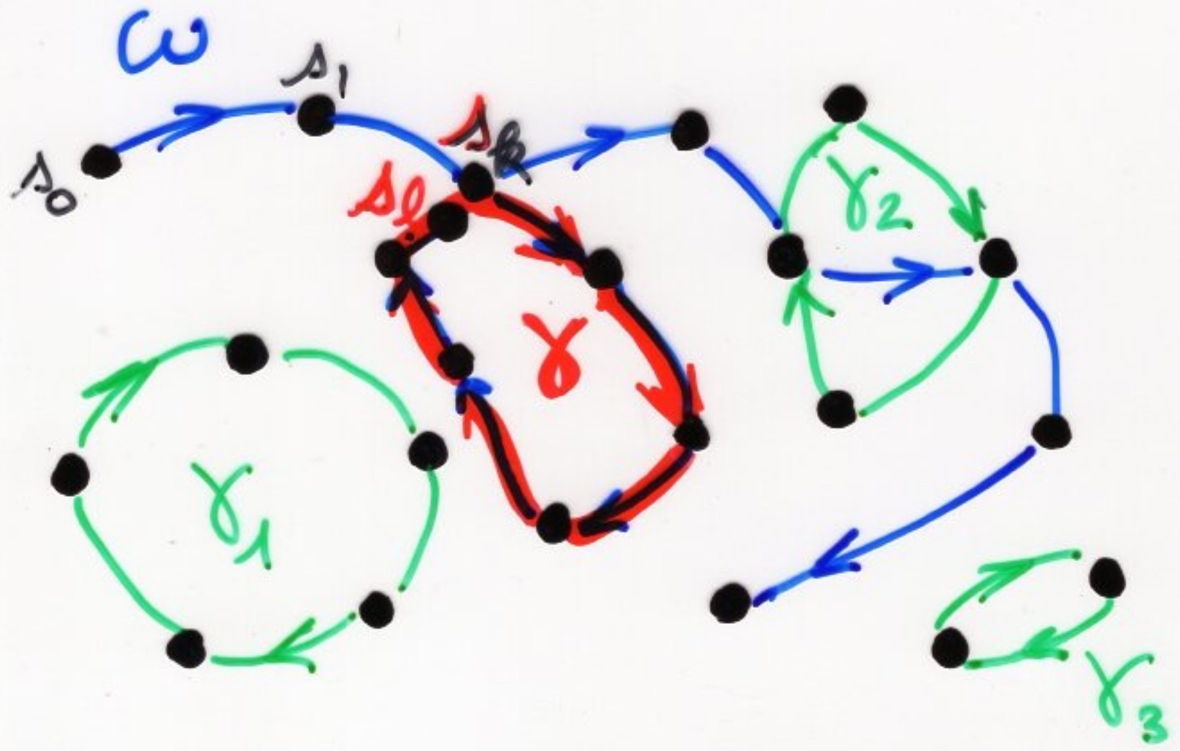
$\gamma = (s_k, s_{k+1}, \dots, s_{\ell-1})$



case (ii) $s_\ell \in \gamma_j = (s_\ell, y_1, \dots, y_p)$

then $\varphi(\xi) = \left(\omega', \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots \gamma_r\}\right)$

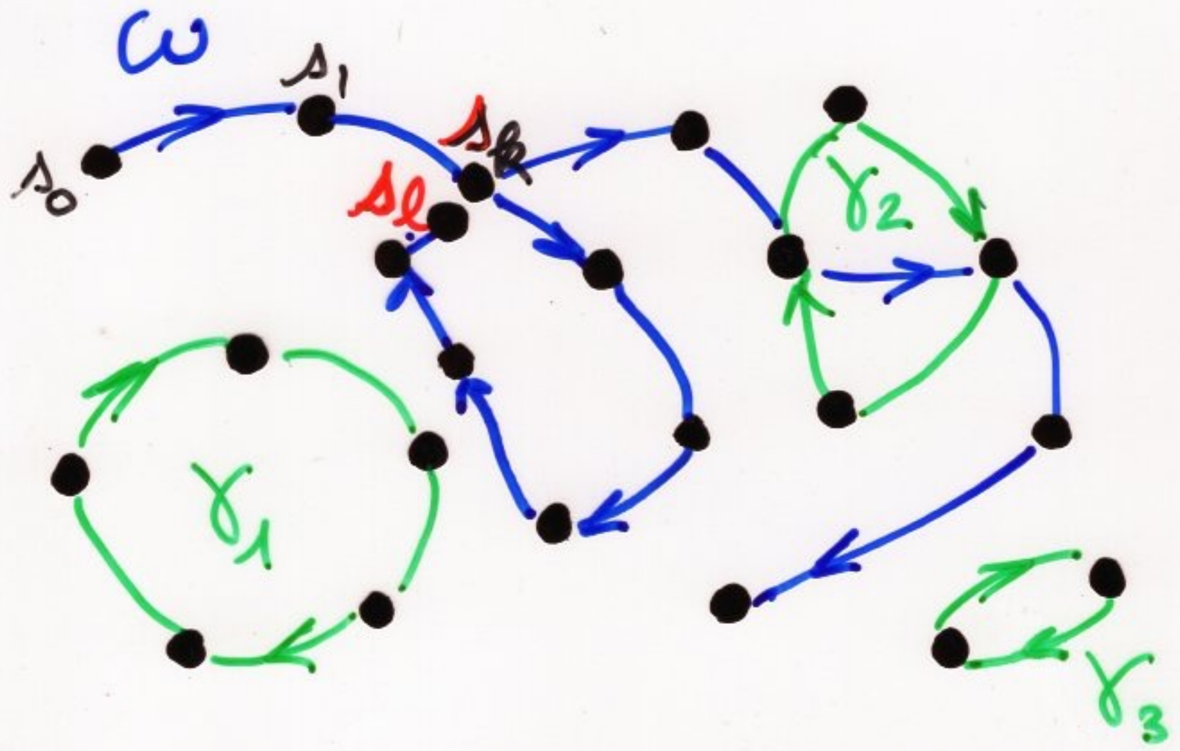
with $\omega' = (s_0, \dots, s_\ell, y_1, \dots, y_p, s_\ell, s_{\ell+1}, \dots, s_n)$



case (ii) $\quad s_\ell \in \gamma_j = (s_\ell, y_1, \dots, y_p)$

then $\quad \varphi(\xi) = \big(\omega' ; \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots \gamma_r\}\big)$

with $\quad \omega' = (s_0, \dots, s_\ell, y_1, \dots, y_p, s_\ell, s_{\ell+1}, \dots, s_n)$



"killing involution" proof

$$\sum_{\alpha \in E^{+}} v(\alpha) - \sum_{\beta \in E^{-}} v(\beta) = \sum_{\gamma \in F} \pm v(\gamma)$$

$$+\alpha \xleftrightarrow{\ \varphi\ } -\beta$$

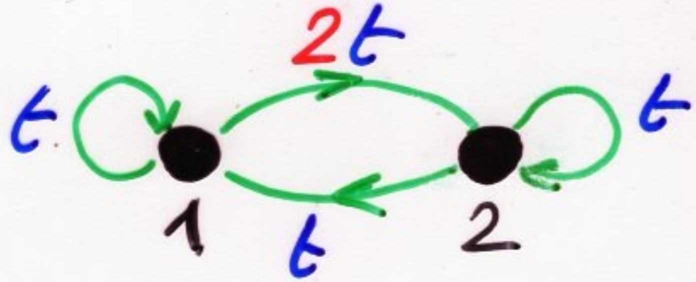
$$v(\alpha) = v(\beta)$$

~~"inclusion-exclusion" proof~~

an example and an exercise

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

example



$S = \{1, 2\}$

$$A = \begin{bmatrix} t & 2t \\ t & t \end{bmatrix}$$

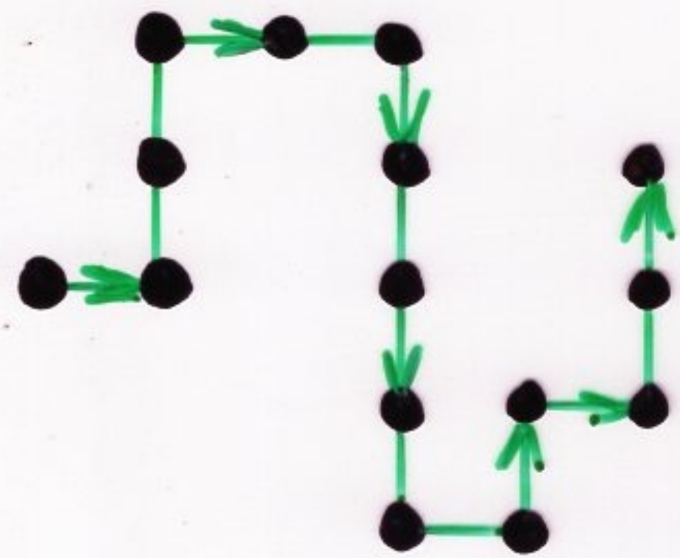
$$D = 1 - (t + t + 2t^2) + t^2$$

$$N_{1,1} = 1 + t$$

$$\sum_{\substack{\omega \\ 1 \rightsquigarrow 1 \\ \text{path}}} v(\omega) = \frac{1+t}{1-2t-t^2}$$

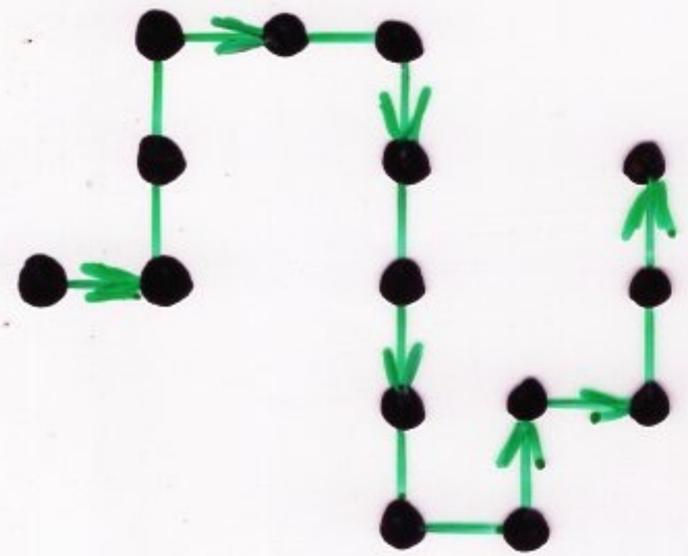
exercise

directed paths
(on a square lattice)



- elementary steps E, N, S
- self-avoiding

prove that the
generating function
(paths enumerated by the length)
is: $$\frac{1+t}{1-2t-t^2}$$



hint: find a bijection with paths on a graph

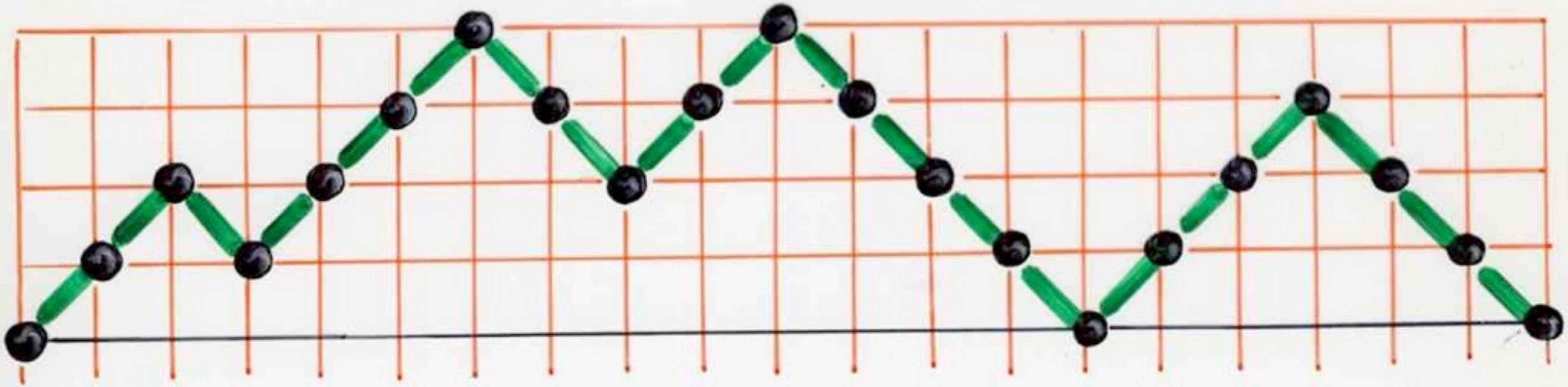
with 3 vertices

another example

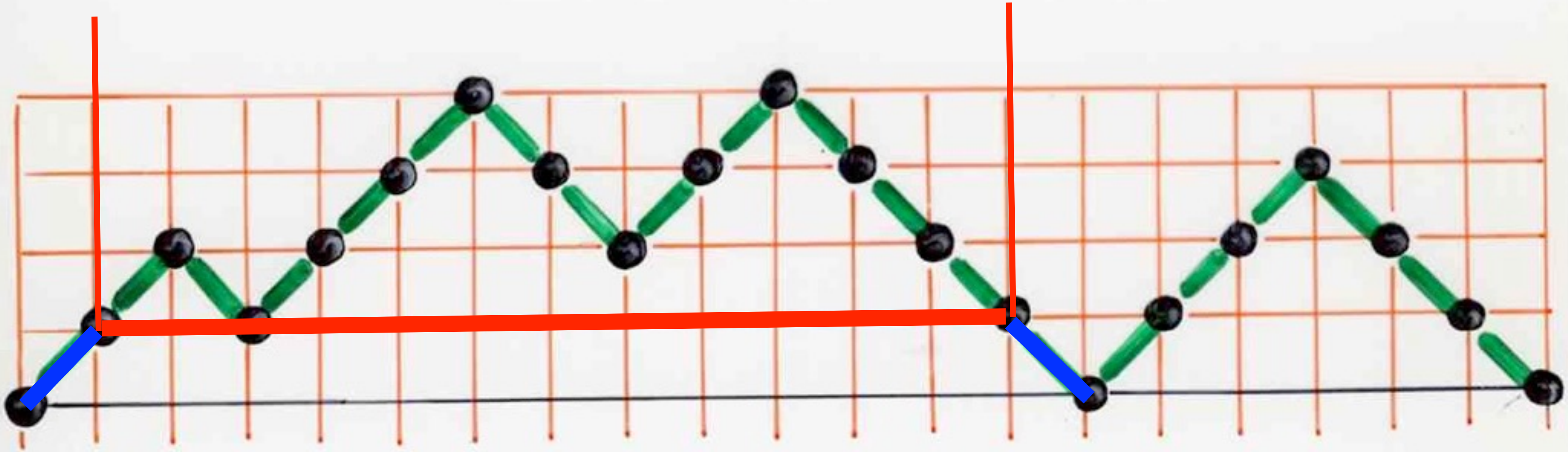
$$\sum_{\substack{\omega \\ \text{path on } S \\ i \leadsto j}} v(\omega) = \frac{N_{i,j}}{D}$$

bounded Dyck paths

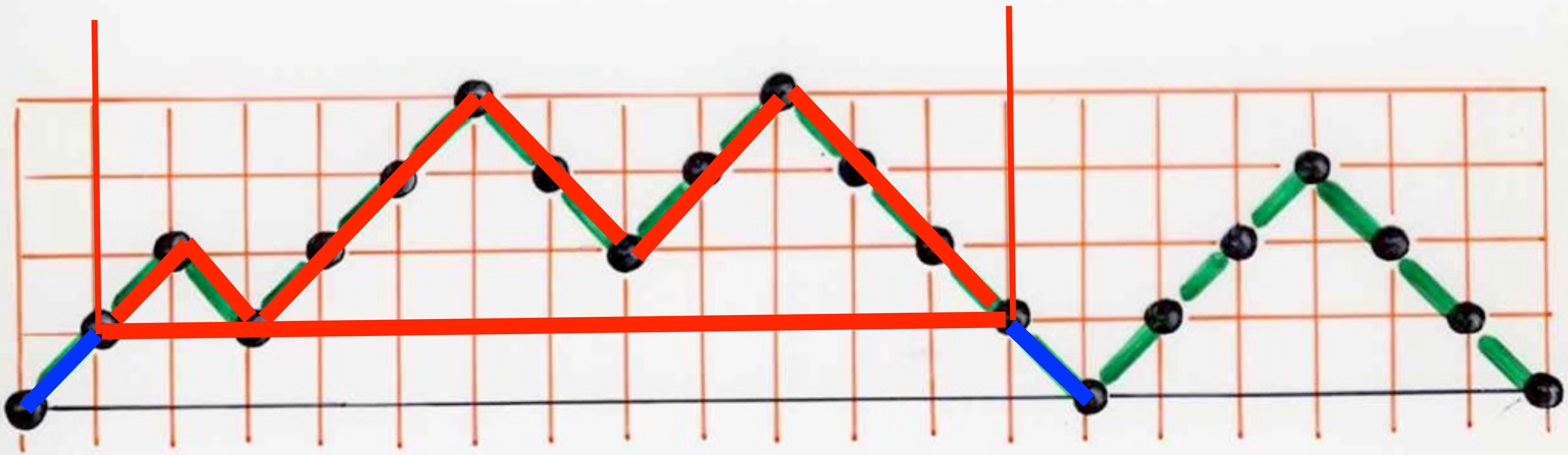
Dyck path



Dyck path

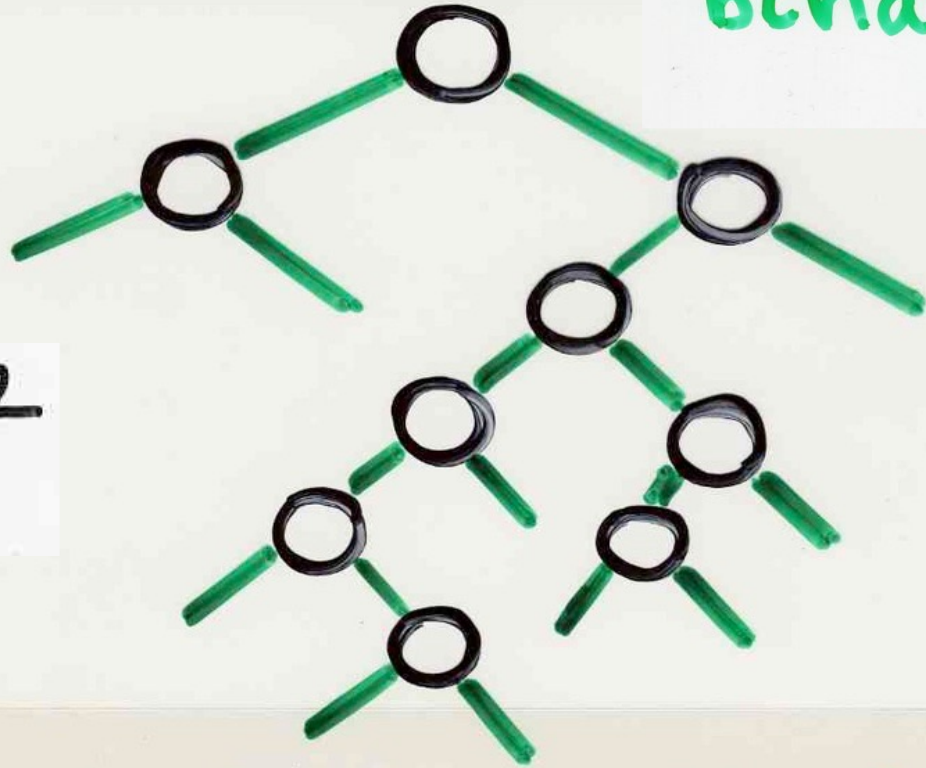


Dyck path



$$D = 1 + tD^2$$

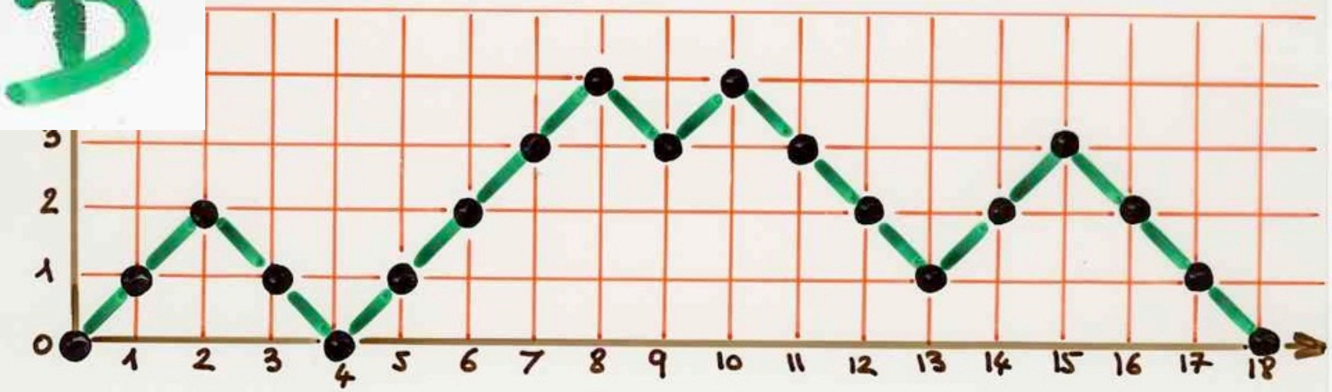
binary tree



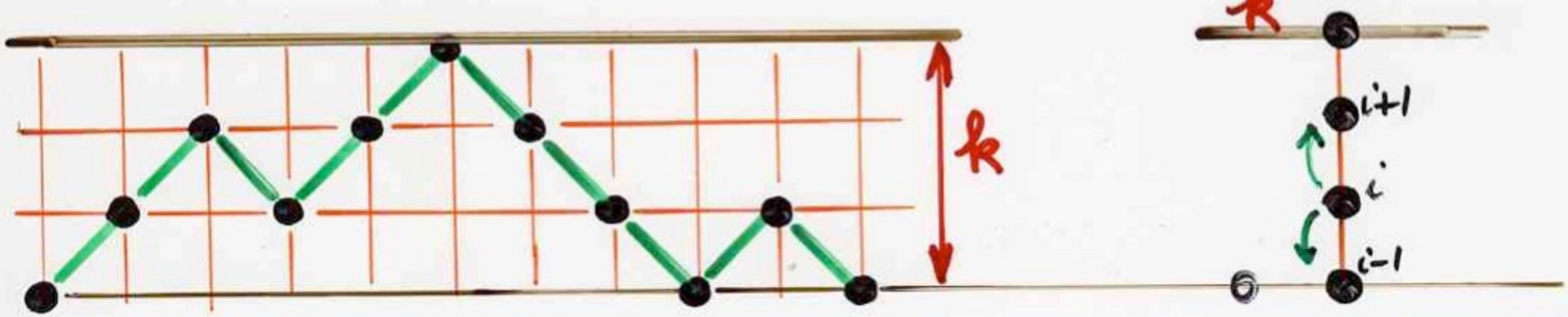
$$A = 1 + tA^2$$

Dyck path

$$D = 1 + tD^2$$



ex : Dyck path
bounded at height k



$$\sum_{\omega} t^{|\omega|/2} = \frac{F_k(t)}{F_{k+1}(t)}$$

Dyck paths
bounded k

$$A = (a_{ij}) = \begin{pmatrix} 0 & t & & \cdots & 0 \\ t & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & t \\ & \cdots & & t & 0 \end{pmatrix}$$

$= n$

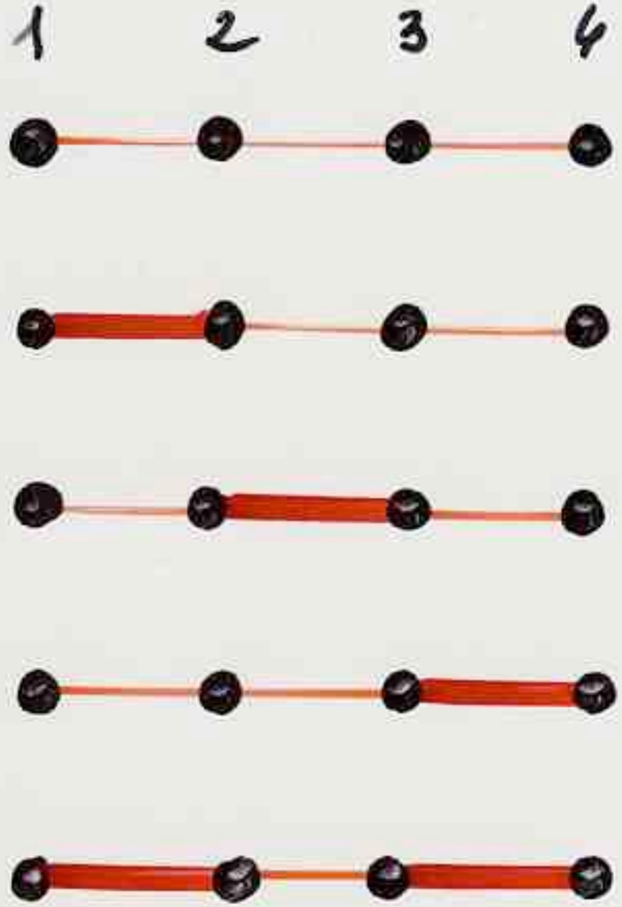
Fibonacci polynomials

$$F_k(x) = \sum_{\substack{M \\ \text{matchings} \\ \text{of } [1,\, k]}} (-x)^{|M|}$$

matching of $[1, k]$

$=$ set of 2 by 2 disjoint edges $(i, i+1)$

(or dimers)



1

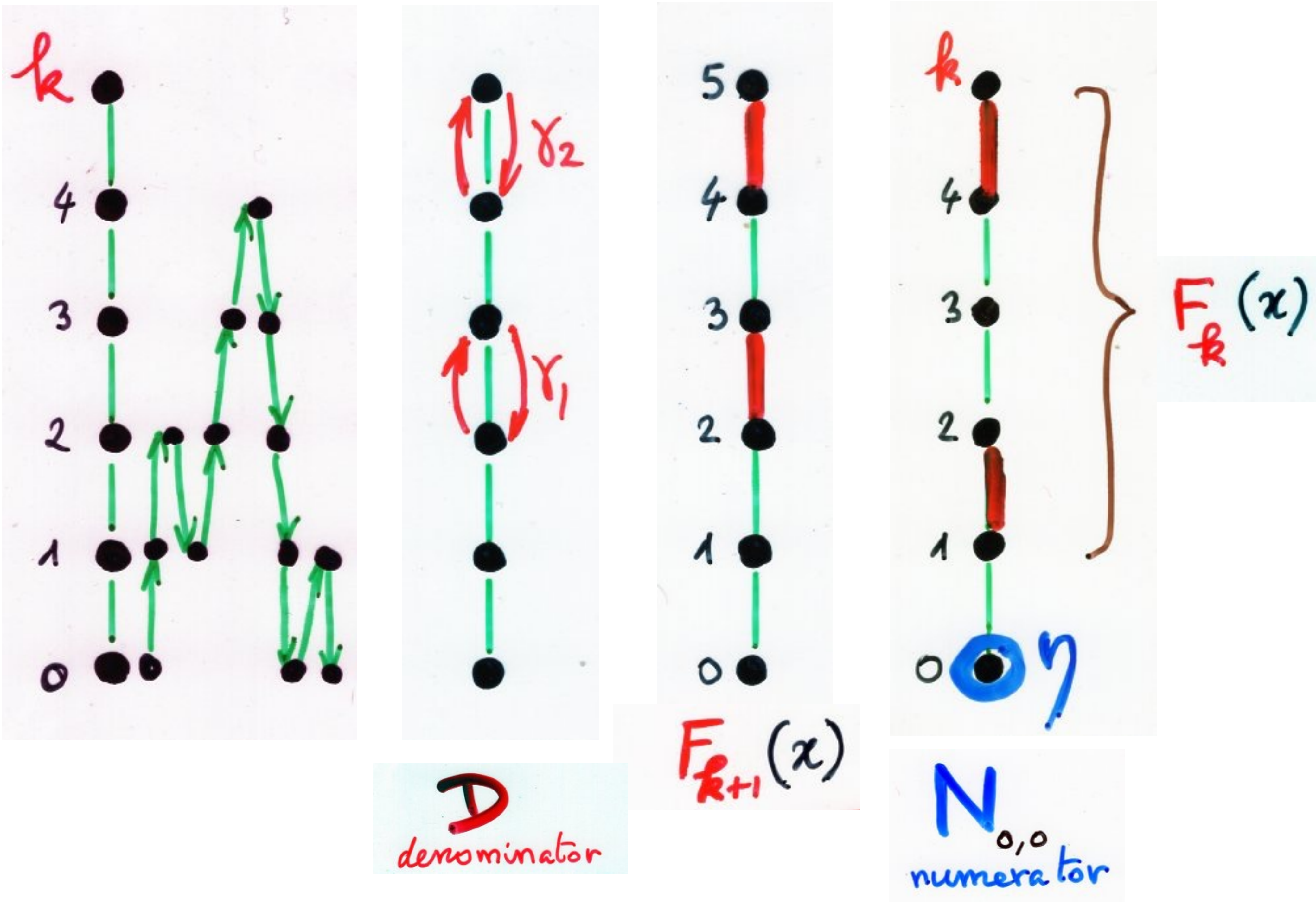
$-x$

$-x$

$-x$

x^2

$$F_4(x) = 1 - 3x + x^2$$


k
4
3
2
1
0
γ2
γ1
D
denominator
5
4
3
2
1
0
F_{k+1}(x)
k
4
3
2
1
0
η
F_k(x)
N_{0,0}
numerator

$$\sum_{\omega} t^{|\omega|/2} = \frac{F_k(t)}{F_{k+1}(t)}$$

Dyck paths

bounded k

Fibonacci polynomials

$= n$

Fibonacci polynomials

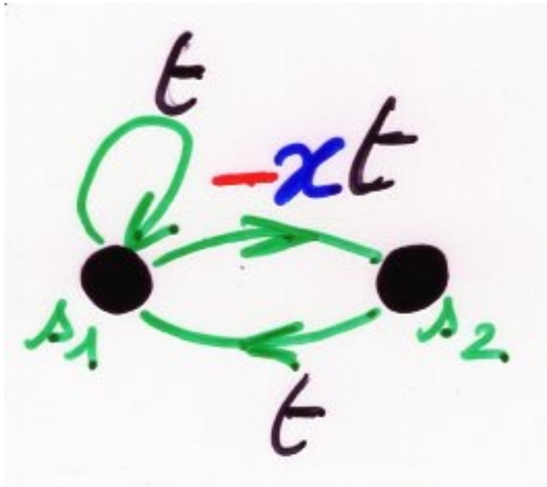
$$F_n(x) = \sum_{k\geq 0} (-1)^k a_{n,k}\, x^k = \sum_{\substack{M \text{ matchings}\\ \text{of } \{1,-,n\}}} (-x)^{|M|}$$

$a_{n,k}$ = number of matchings of $\{1,2,\dots,n\}$ with k dimers

$= n$

bijection

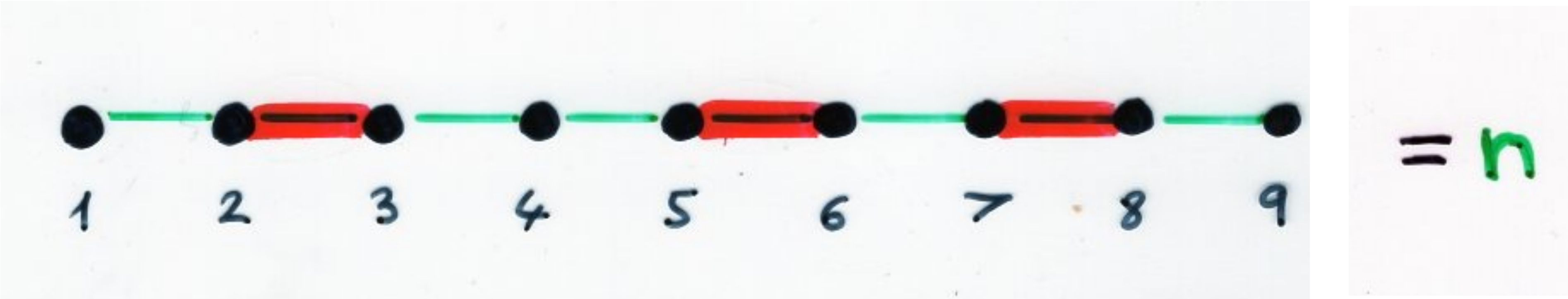
matchings of $[1, n]$ $\longleftrightarrow$ paths ω length n going from s_1 to s_1

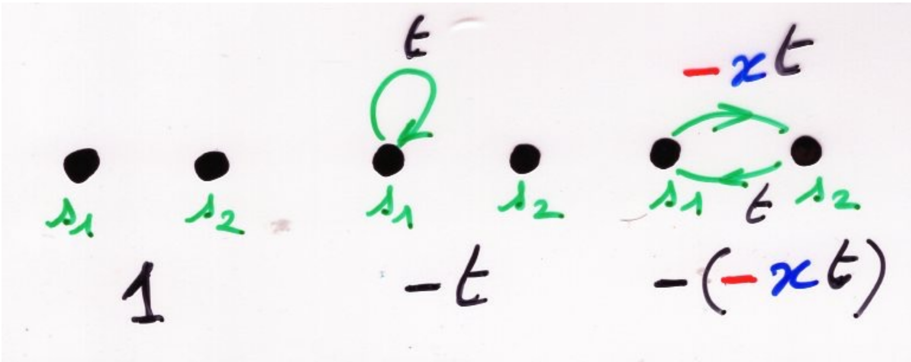


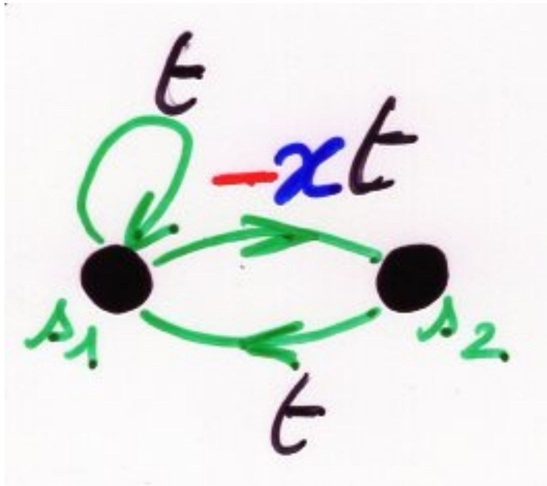
such that

$$v(\omega) = (-x)^k t^n$$

k = number of dimers of the matching







$$\sum_{n\geq 0} F_n(x)\,t^n = \frac{1}{1-t+xt^2}$$

Fibonacci polynomials

$$F_{n+1}(x) = F_n(x) - x F_{n-1}(x)$$

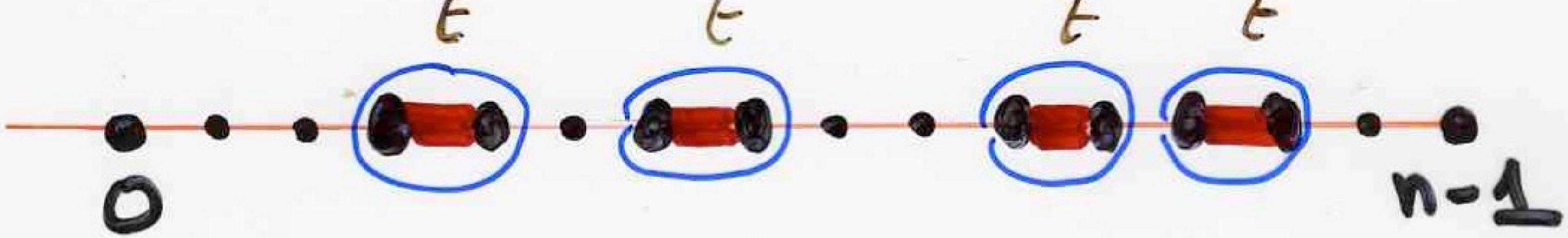
$$F_0 = F_1 = 1$$

1

$1 - t$

$1 - 2t$

$1 - 3t + t^2$

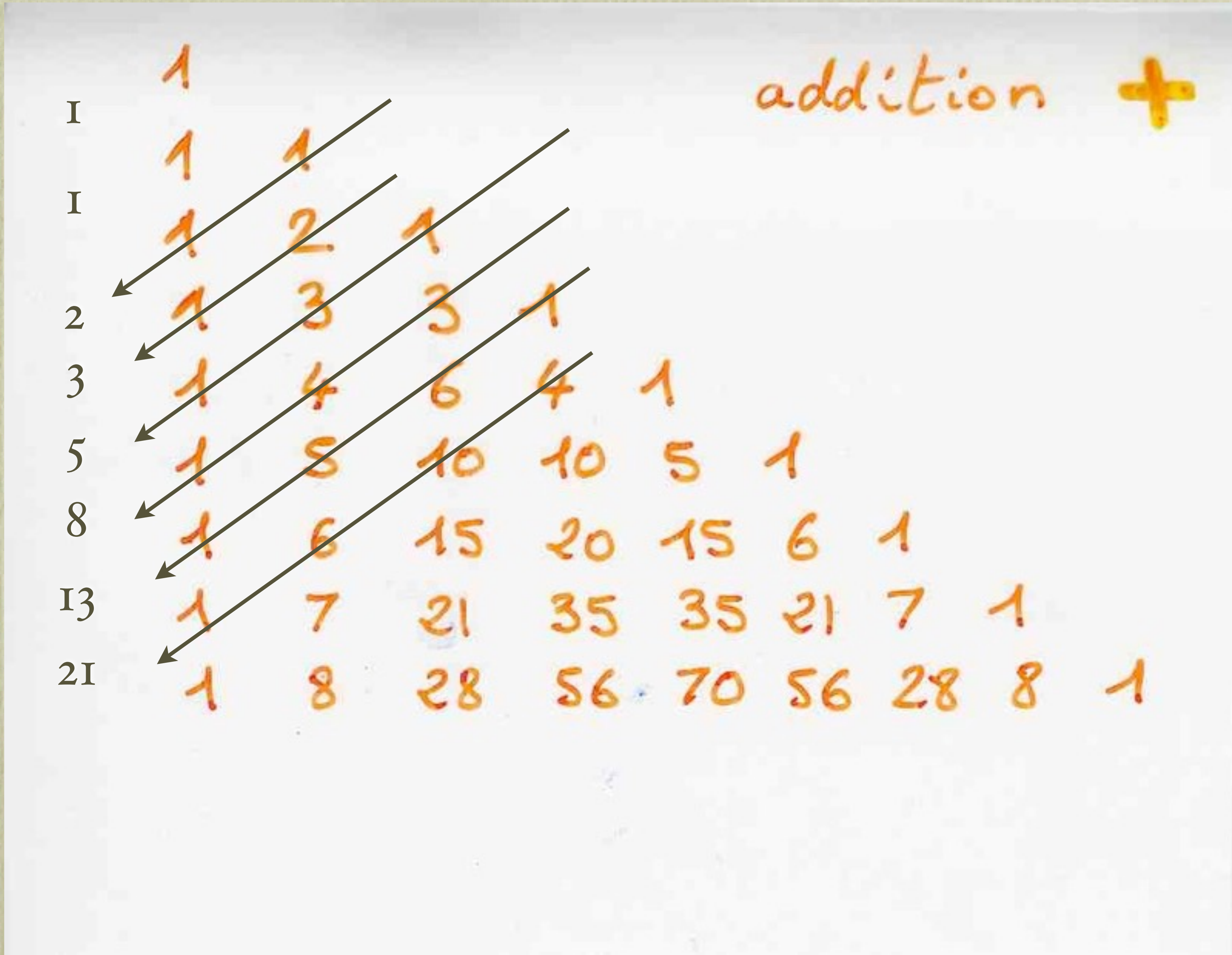


trivial heap of dimers

exercise

$a_{n,k}$ = number of matchings of $\{1, 2, \ldots, n\}$ with k dimers

$$a_{n,k} = \binom{n-k}{k}$$


addition +
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1
1
2
3
5
8
13
21

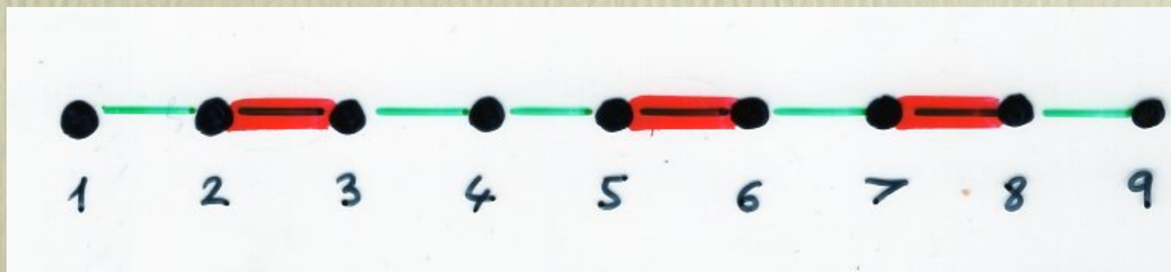
Pingala (2nd century B

Pingala Laghu (short syllabe)
Guru (long syllabe)

two classes of meters in Sanskrit

- Akṣarachandaḥ
  Chandaḥ number of syllables
  later 4 feet (pāda)
- number of mātrās (time measure)
  short syllabe : one mātrās
  long syllabe : two mātrās

relation with Fibonacci numbers ?

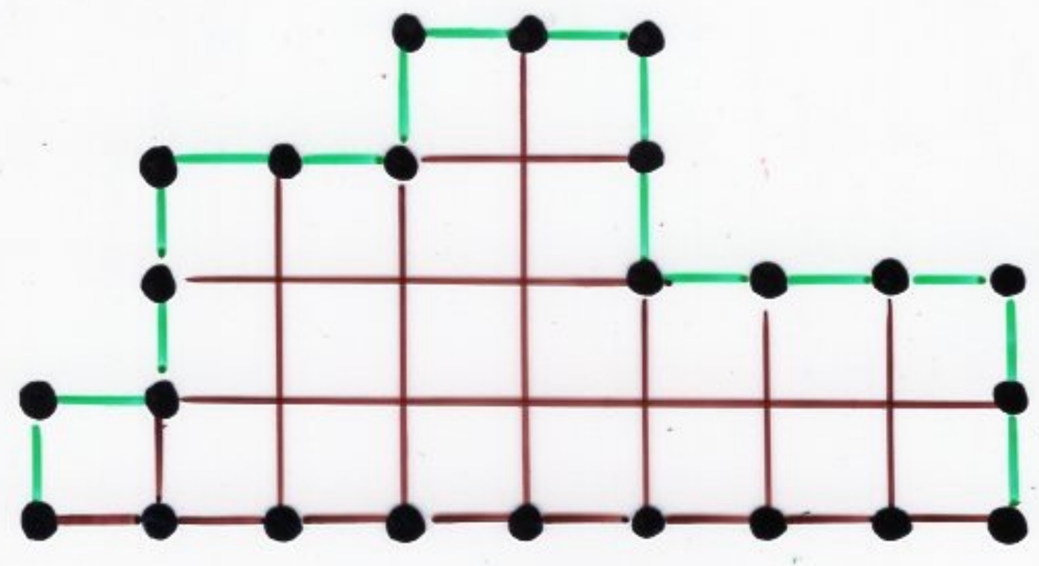


exercise

Fibonacci and polyominoes

exercise

stack
polyomino



- prove that the number of stack polyominoes with perimeter $2n+4$ is F_{2n} (Fibonacci number)

(with a bijection

stack polyominoes length $2n+4$ $\longleftrightarrow$ words of $(x+aa)^*$ of length $2n$

$\left(= \text{matchings of } [1, 2n]\right)$

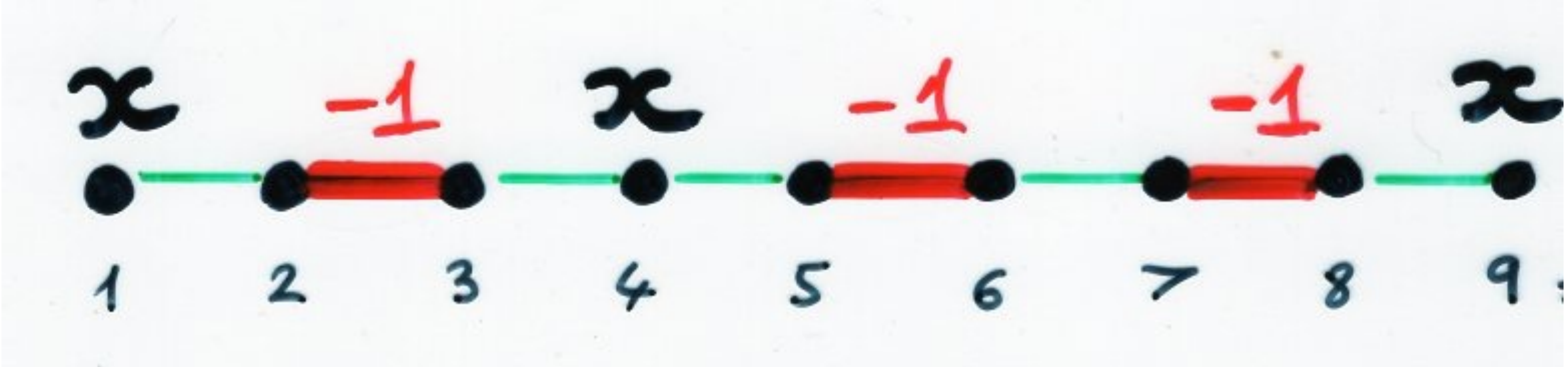
Fibonacci
and
Tchebychef polynomials

$$\sin\big((n+1)\theta\big) = \sin\theta \; U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind

sequence of orthogonal polynomials

$$\frac{2}{\pi}\int_{-1}^{+1} U_n(x)\, U_m(x)\,(1-x^2)^{1/2}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{else} \end{cases}$$



$$M_n(x) = \sum_{k \geqslant 0} (-1)^k a_{n,k}\, x^{n-2k}$$

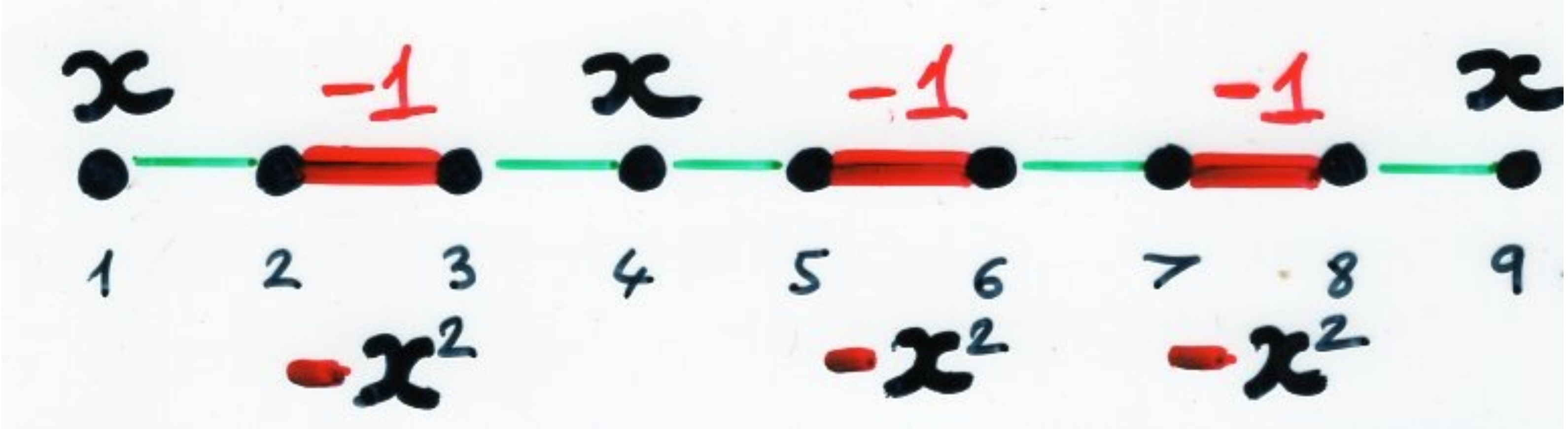
matching polynomial of the segment graph

$$= \sum_{M} (-1)^{|M|}\, x^{ip(M)}$$

M matchings of $\{1, .., n\}$

$ip(M)$ is the number of isolated points of M

$a_{n,k}$ = number of matchings of $\{1,2,\dots,n\}$ with k dimers



$$M_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k}\, x^{n-2k}$$

matching polynomial of the segment graph

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1,..,n\}}} (-1)^{|M|}\, x^{ip(M)}$$

$ip(M)$ is the number of isolated points of M

$$M_n^*(x) = x^n M_n(1/x)$$

reciprocal

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1,..,n\}}} (-x^2)^{|M|}$$

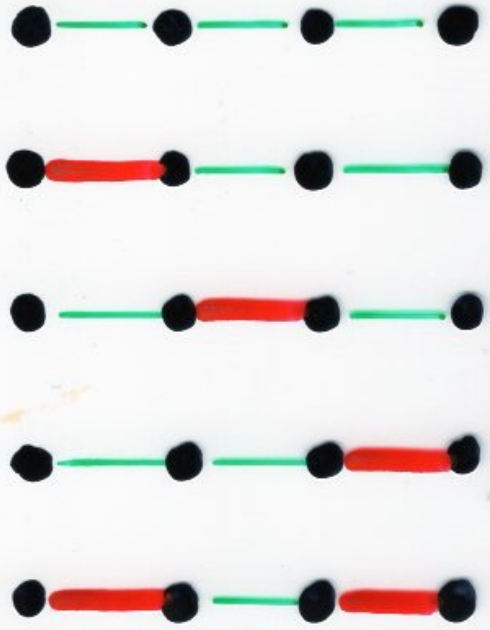
$$= F_n(x^2)$$

$$\sin\left((n+1)\theta\right) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

$$U_n(x) = M_n(2x)$$

same 3-terms recurrence relation



$$x^4 \qquad (2\cos\theta)^4$$

$$-3x^2 \qquad -3(2\cos\theta)^2$$

$$+1 \qquad 1$$

$$\sin 5\theta = \sin\theta\,(16\cos^4\theta - 12\cos^2\theta + 1)$$

Lucas
and
Tchebycheff polynomials

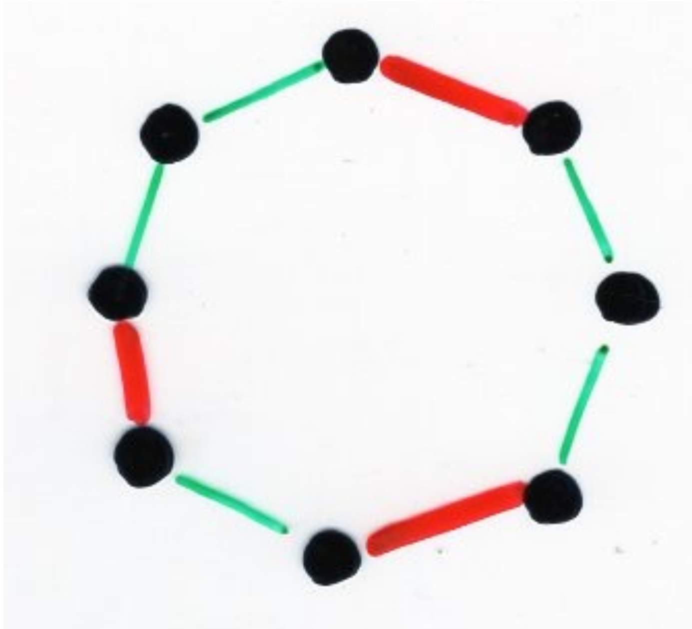
$$\cos(n\theta) = T_n(\cos\theta)$$

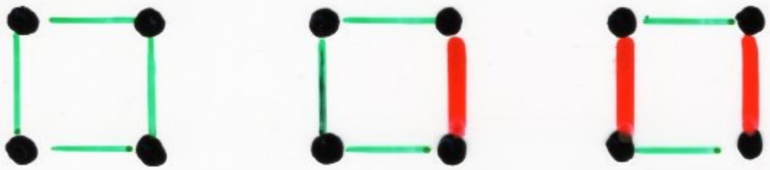
$T_n(x)$

Tchebychef
polynomial
1st kind

$L_n(t)$ Lucas polynomial

$$L_n(x) = \sum_{\substack{\text{matchings } M \\ \text{of a cycle } \gamma \\ \text{length } n}} (-x)^{|M|}$$





1   4   2

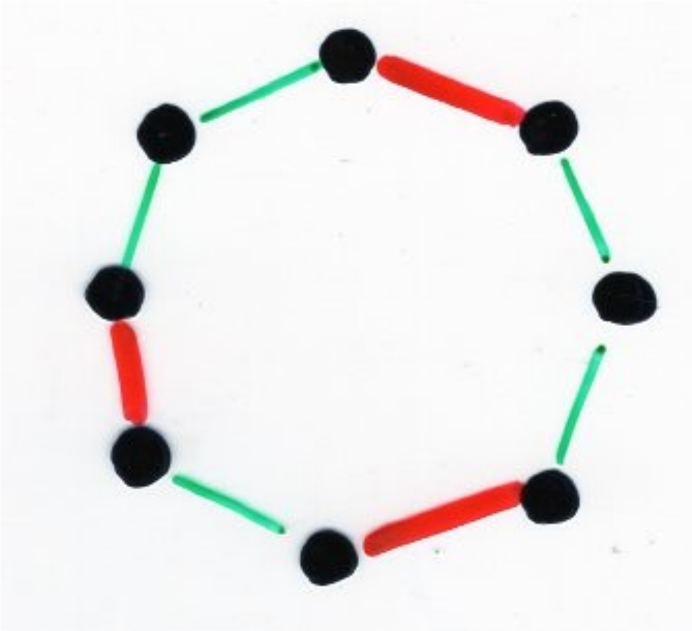
$$L_4(x) = 1 - 4x + 2x^2$$

reciprocal of $L_n(x^2)$ is

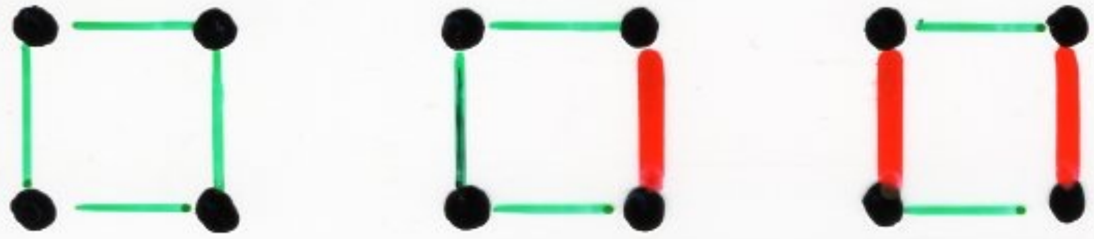
$$C_n(x) = \sum_{\substack{\text{matching } M \\ \text{of } \gamma}} (-1)^{|M|} x^{ip(M)}$$

number of isolated points

matching polynomial of the cycle graph



$$T_n(x) = \frac{1}{2} C_n(2x)$$



1 4 2

$$L_4(x) = 1 - 4x + 2x^2$$

$$L_4(x^2) = 1 - 4x^2 + 2x^4$$

$$L_4^*(x^2) = x^4 - 4x^2 + 2$$

$$C_4(x) = 16x^4 - 16x^2 + 2$$

$$\left(8\cos^4\theta - 8\cos^2\theta + 1\right) = \cos 4\theta$$

$$T_n(\cos\theta)$$

same 3-terms
recurrence
relation

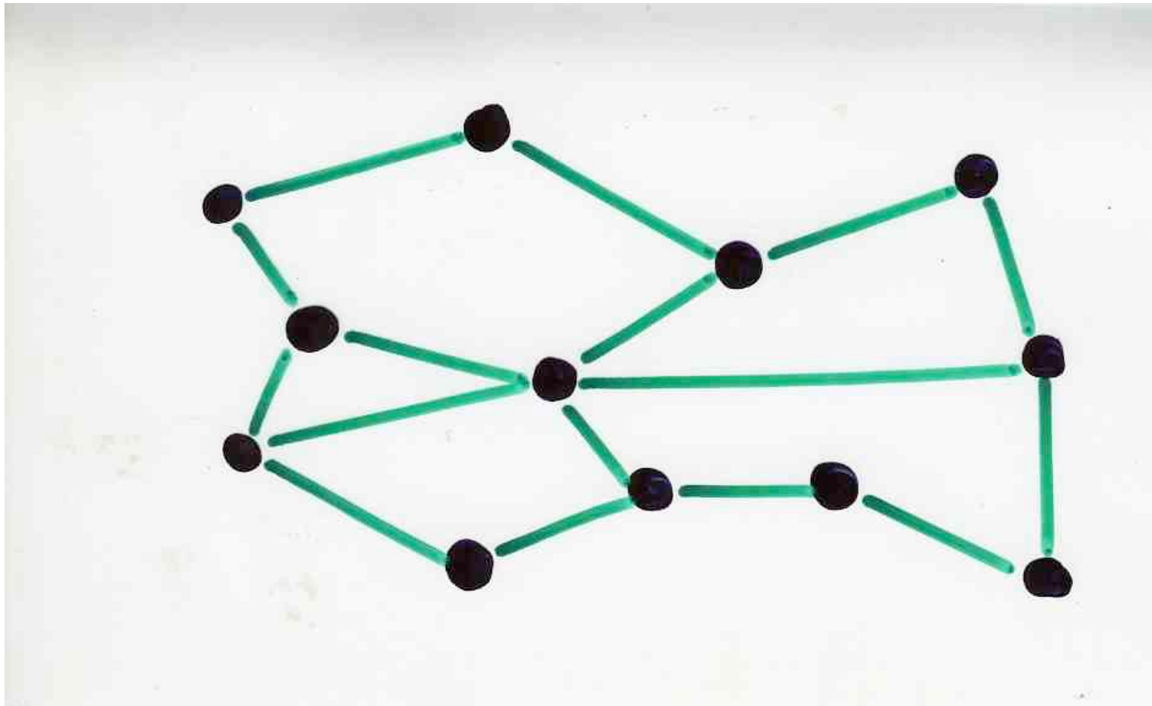
exercise

factorisation

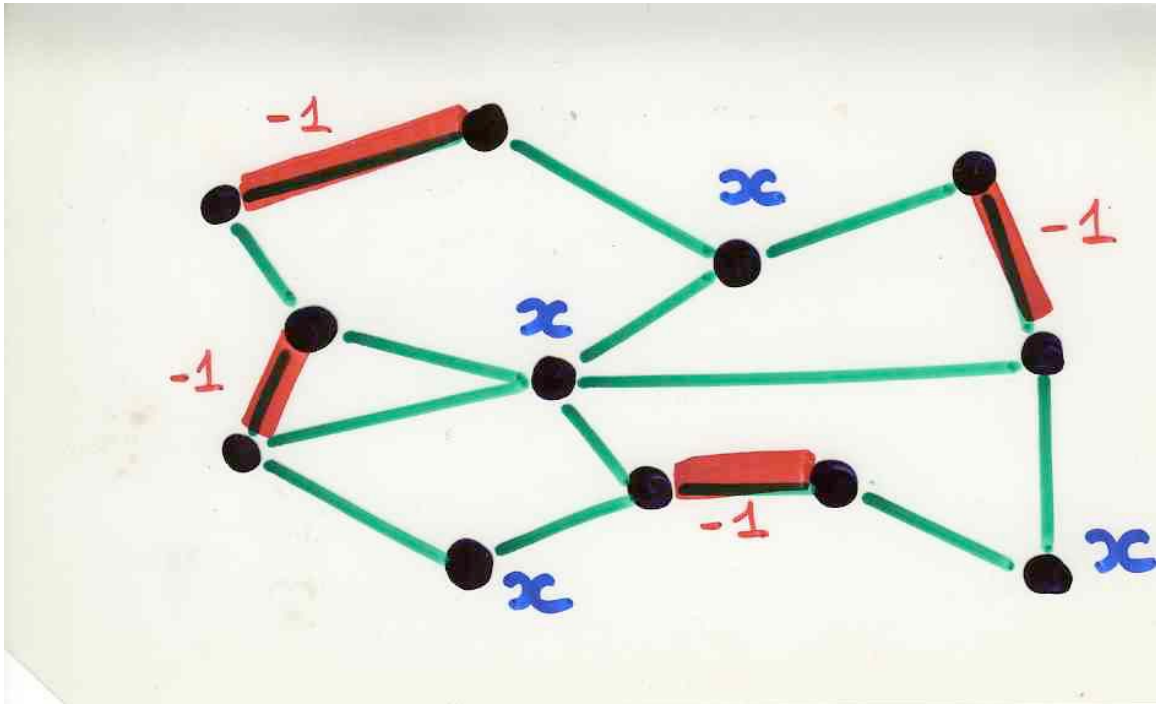
$$F_{2n+1}(t) = F_n(t) \times L_{n+1}(t)$$

complement

Matching polynomial of a graph

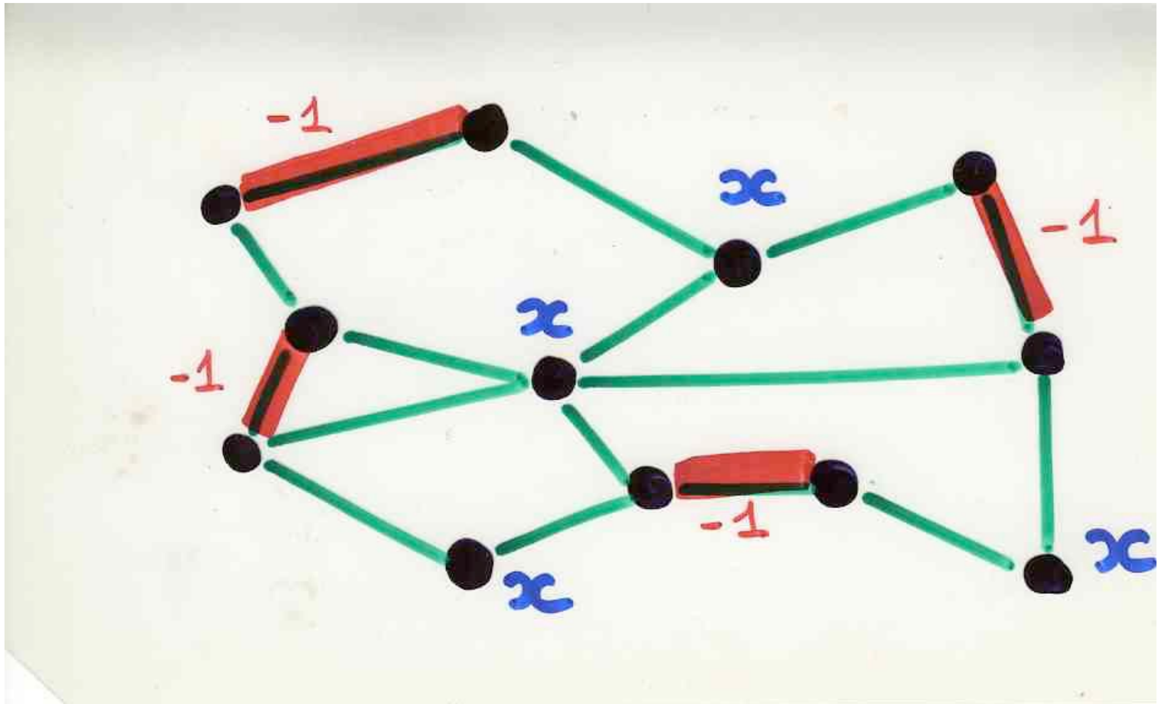


matching of a graph G = set of 2 by 2 disjoint edges



matching of a graph G = set of 2 by 2 disjoint edges

perfect matching (or 1-factor) = no isolated point



Matching polynomial of a graph G

$$M_G(x) = \sum_{\substack{\text{matchings } M \\ \text{of } G}} (-1)^{|M|} x^{ip(M)}$$

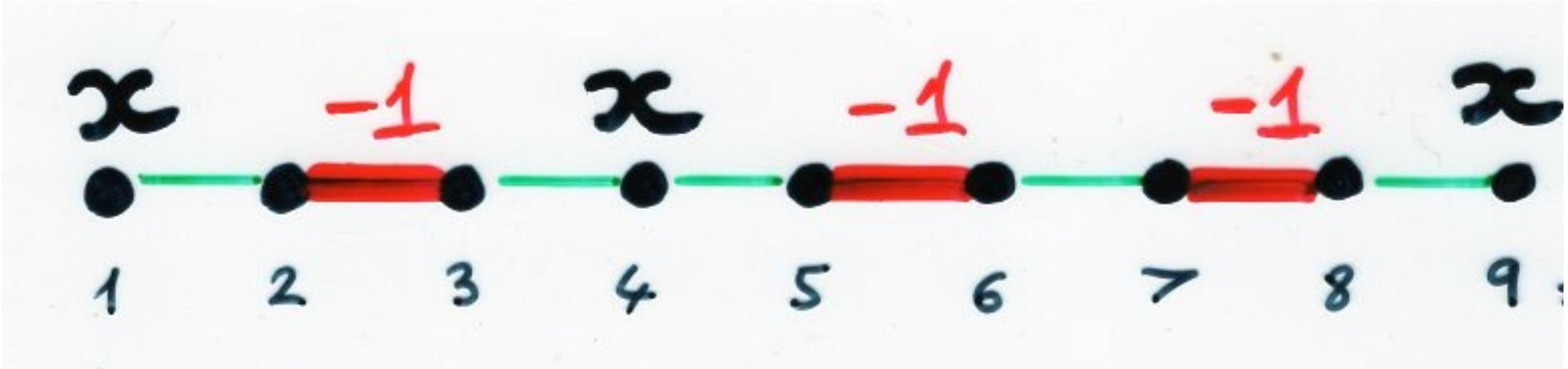
isolated points

$$= \sum_{M} (-1)^{|M|} x^{n-2|M|}$$

n = nb of vertices of G

Prop For every graph G the zeros of the matching polynomial are real numbers

→ course on heaps of pieces



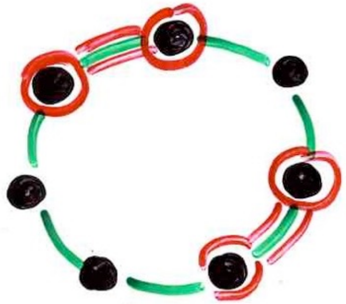
$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

$$U_n(x) = M_n(2x)$$

$$\cos(n\theta) = T_n(\cos\theta)$$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

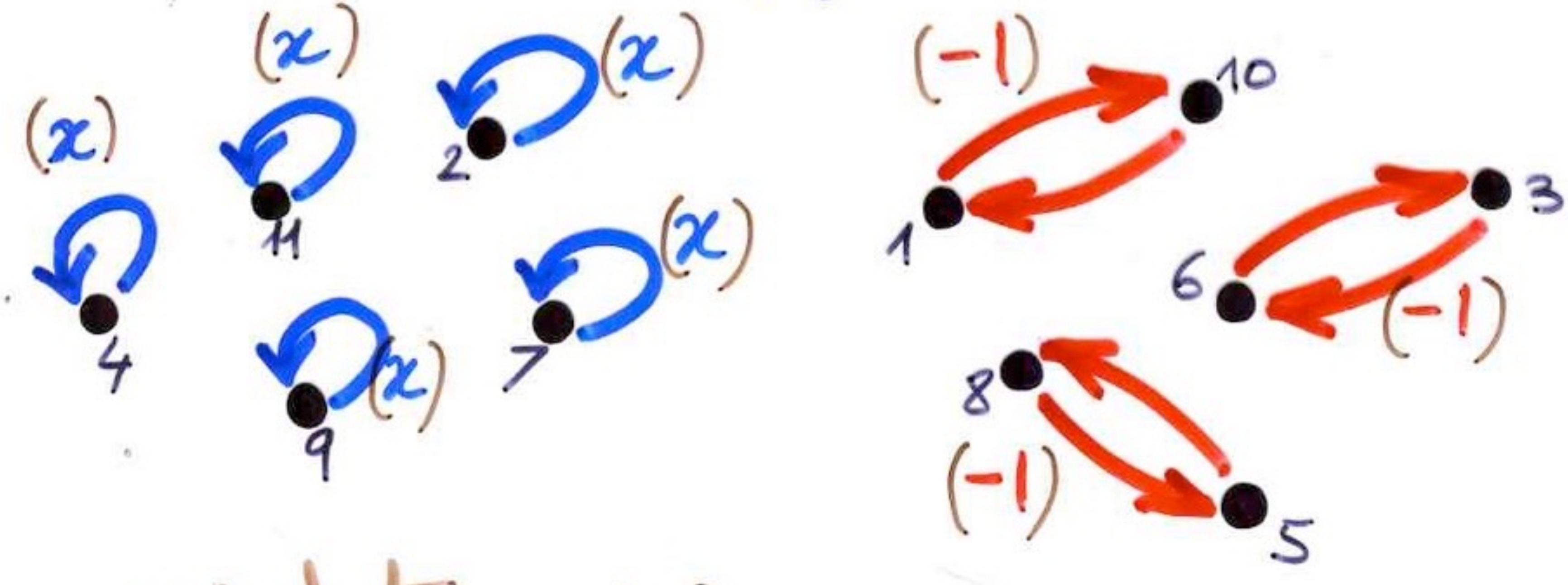


$T_n(x)$ Tchebychef polynomial 1st kind

complete graph matching polynomial = Hermite polynomial

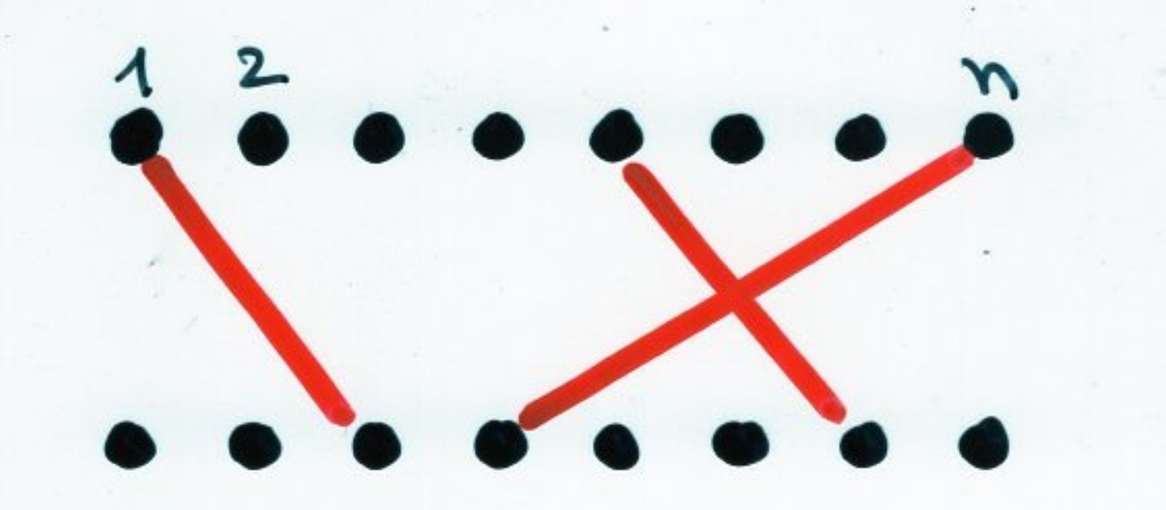
$H_n(x)$

Hermite configurations



weight (x)

(-1)



complete bipartite graph

matching polynomial = Laguerre polynomial $L_n(x)$

<u>exercise</u> give a formula for the coefficient of $H_n(x)$, $L_n(x)$

Hermite polynomial

example

$$\sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{i,j}}{D}$$

back to Strahler numbers

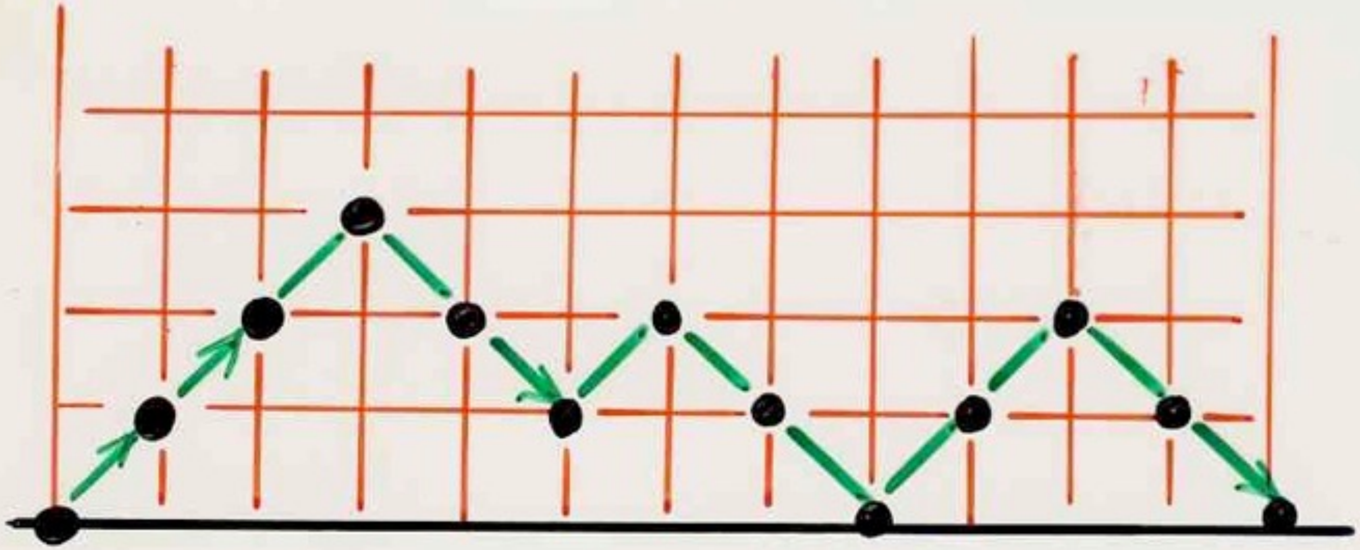
$$S(t,x) = \sum_{k\geq 0} S_k(t)\, x^k$$

$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

Françon (1984)

Knuth (2005)



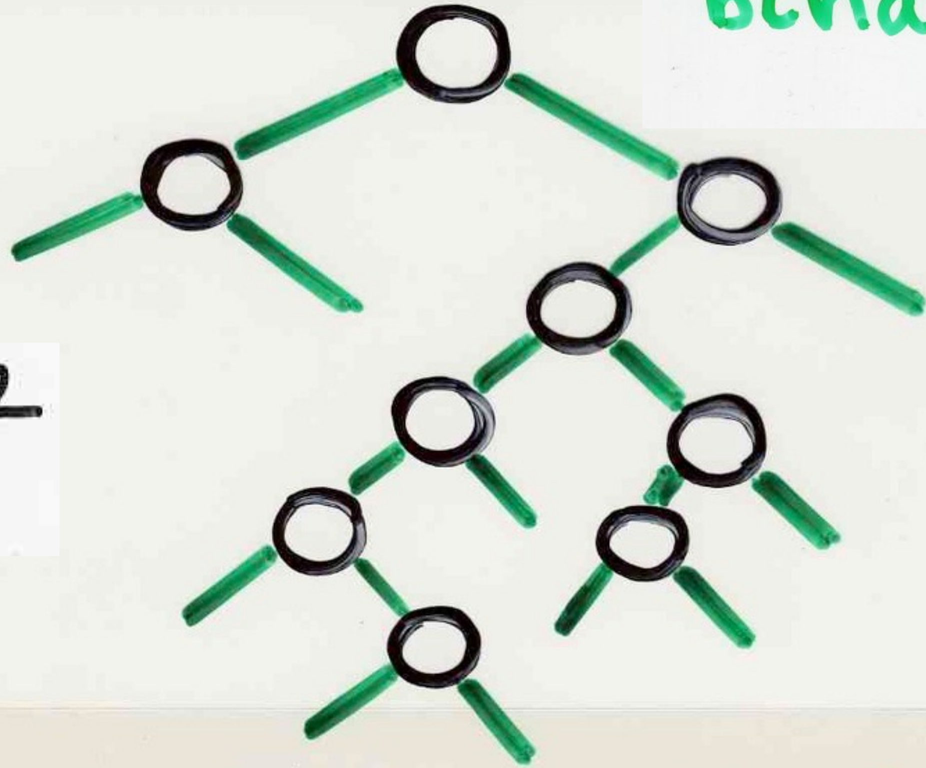
length

$2n = 12$

Dyck path

C_n = number of Dyck path of length $2n$

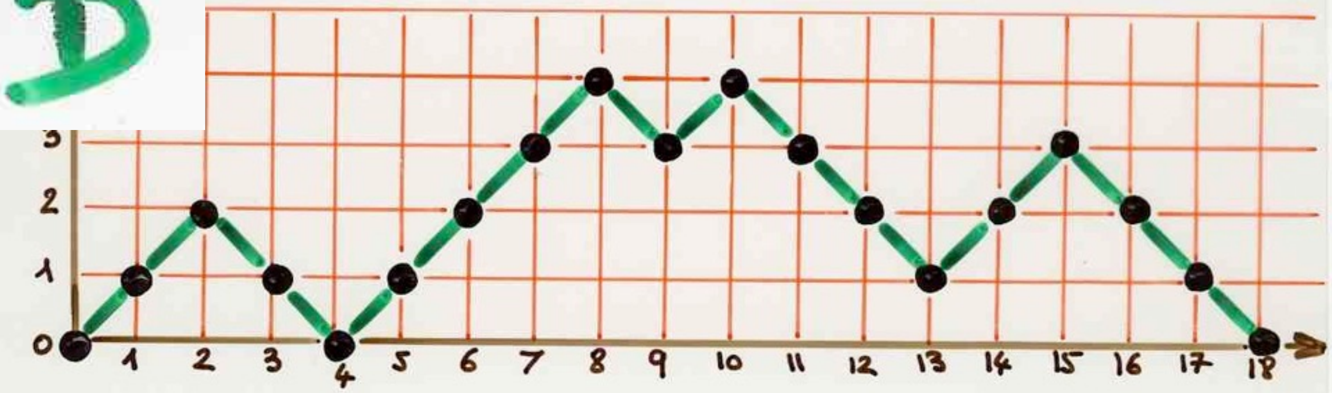
binary tree



$$A = 1 + tA^2$$

Dyck path

$$D = 1 + tD^2$$



Dyck path w

Height $h(w)$

logarithmic height $lh(w)$

$$= \lfloor \log_2 (1+h(w)) \rfloor$$

$$lh(w) = k$$

$$\Longleftrightarrow \quad 2^{k}-1 \leq h(w) < 2^{k+1}-1$$

$$D(t,x) = \sum_{n,k} D_{n,k}\, x^k t^n$$

number of Dyck paths ω
with length $|\omega| = 2n$
and logarithmic height
$lh(\omega) = k$

$D(t,x)$ satisfies the same functional equation than $S(t,x)$

enumerating binary trees according to the number of internal vertices (t) and Strahler number (x)

$\longrightarrow$ proof in chapter 3

$$S(t,x) = \sum_{k\geq 0} S_k(t)\, x^k$$

$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

Françon (1984)

Knuth (2005)

(complete)

binary trees $\xleftrightarrow[(1984)]{\text{Françon}}$ Dyck paths

n (internal) vertices — length $2n$

Strahler nb $= k$ — log. height $lh(w) = k$

$\xleftrightarrow[(2005)]{\text{Knuth}}$

$$S_{\leq k}(t) = \frac{F_{2^k-2}(t)}{F_{2^k-1}(t)} \qquad (k \geq 2)$$

generating function
for binary trees
with Strahler number $\leq k$ $\quad = k$

Pascal triangle

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1
1	10	45	120	210	252	210	120	45	10
1	11	55	165	330	462	462	330	165	55
1	12	66	220	495	792	924	792	495	220
1	13	78	286	715	1287	1716	1716	1287	715
1	14	91	364	1001	2002	3003	3432	3003	2002
1	15	105	455	1365	3003	5005	6435	6435	5005

$$S_{\leq k}(t) = \frac{F_{2^k-2}(t)}{F_{2^k-1}(t)} \qquad (k \geq 2)$$

generating function for binary trees with Strahler number $\leq k$ $= k$

$$S_k(t) = S_{\leq k}(t) - S_{\leq (k-1)}(t) \qquad (k \geq 2)$$

$$= \frac{t^{(2^{k-1}-1)}}{F_{2^k-1}(t)}$$

factorisation

$$F_{2n+1}(t) = F_n(t) \times L_{n+1}(t)$$

$$n = 2^k - 1 \qquad \frac{n+1}{2} = 2^{k-1}$$

$$2n+1 = 2^{k+1} - 1$$

$$F_{2^{k+1}-1}(t) = Z_1(t) Z_2(t) \cdots Z_k(t)$$

$$Z_i(t) = L_{2^i}(t)$$

exercise

proof by « killing involution »

Euler's pentagonal theorem

a_n = number of partitions of n
with distinct parts

$$\sum_{n\geq 0} a_n q^n = \prod_{i\geq 1}(1+q^i)$$

$$\prod (1-q^i) \quad ?$$

EVOLVTIO PRODVCTI INFINITI

$$(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$$

IN SERIEM SIMPLICEM.

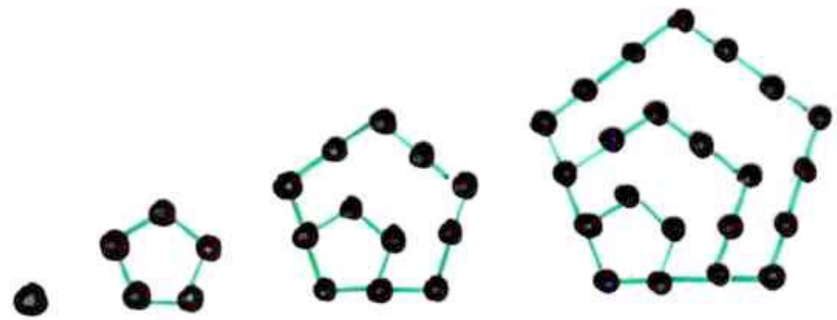
Auctore

L. EVLERO.

$$\prod_{i\geq 1}(1-q^i) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

$$= \sum_{n\geq 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right)$$

Euler pentagonal identity



1 5 12 22 ...

pentagonal numbers $\frac{n(3n-1)}{2}$

The coefficient of q^n in $\prod_{i \geq 1} (1 - q^i)$

$=$ (number of partitions of n into an even number of distinct parts)

$-$ (number of partitions of n into an odd number of distinct parts)

find a proof by of Euler pentagonal theorem with the construction of a « killing involution »