An introduction to

enumerative
algebraic combinatorics
bijective

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.xavierviennot.org

IMSc
January-March 2016

Chapter 1
Ordinary generating functions
(4)

IMSc

19 January 2016

From the previous lecture

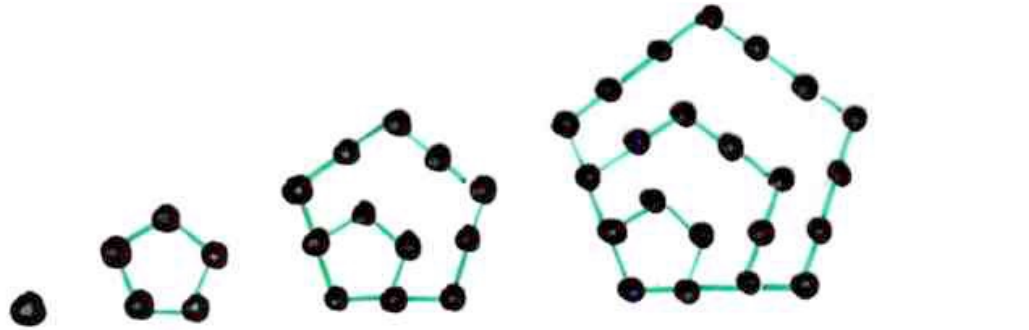
14 January 2016

Euler's pentagonal theorem

$$\prod_{i \geq 1} (1 - q^i) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

$$= \sum_{n \geq 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right)$$

Euler pentagonal identity



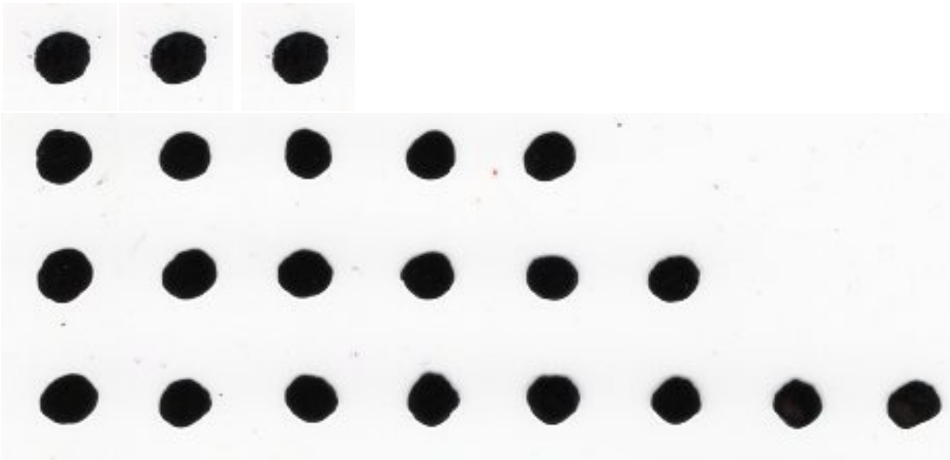
1 5 12 22 . . .

pentagonal numbers $\dfrac{n(3n-1)}{2}$

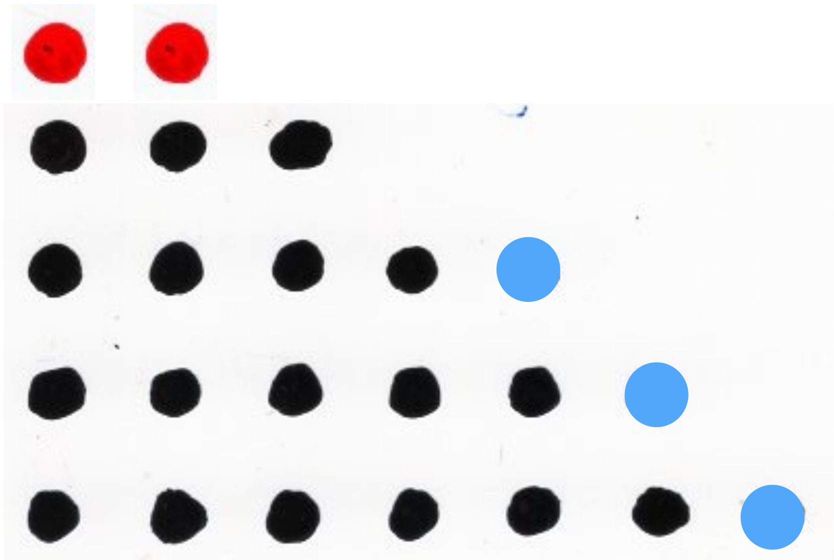
The coefficient of q^n in $\prod_{i\geq 1}(1-q^i)$

$$= \left(\begin{array}{l}\text{number of partitions of } n \text{ into} \\ \text{an even number of distinct parts}\end{array}\right) - \left(\begin{array}{l}\text{number of partitions of } n \text{ into} \\ \text{an odd number of distinct parts}\end{array}\right)$$

Euler pentagonal identity

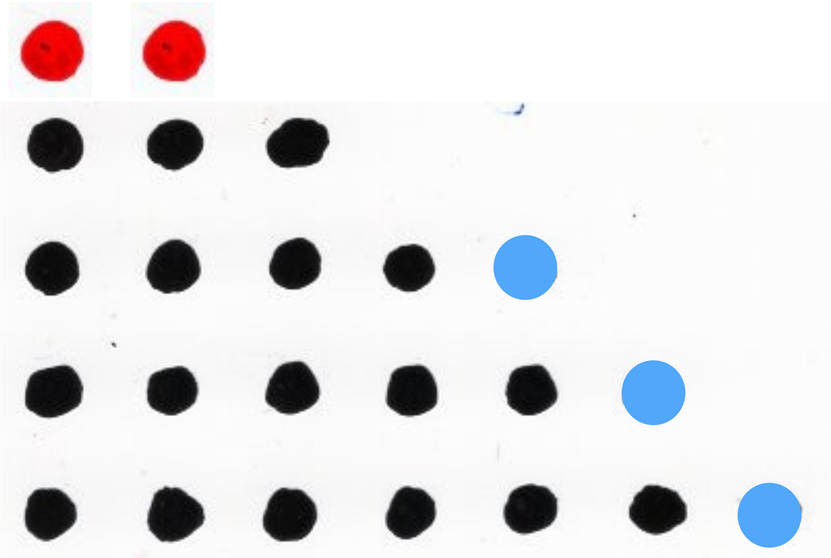


construction of an involution
changing the parity
of the number of rows
of the Ferrers diagram F



$r(F)$ = number of cells of the top row of F

$d(F)$ = longest sequence of cells in diagonal position on the NE border of F

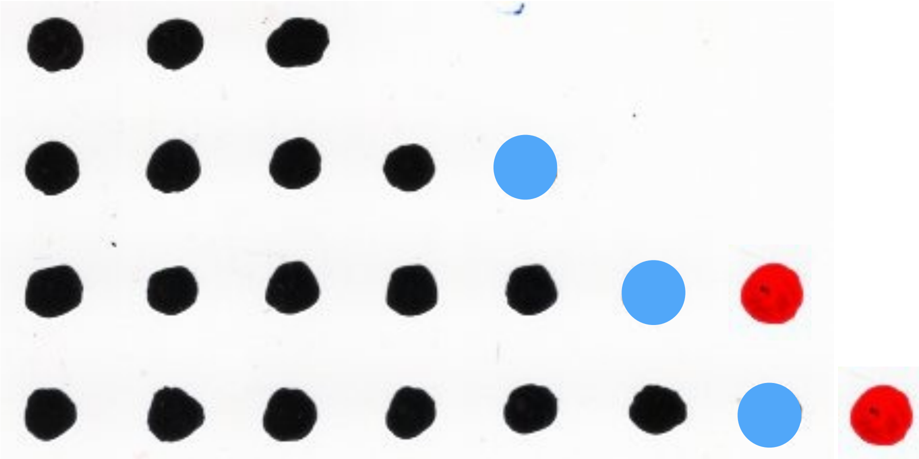


(i) $r(F) \leq d(F)$

involution φ

$r(F)$ = number of cells of the top row of F

$d(F)$ = longest sequence of cells in diagonal position on the NE border of F

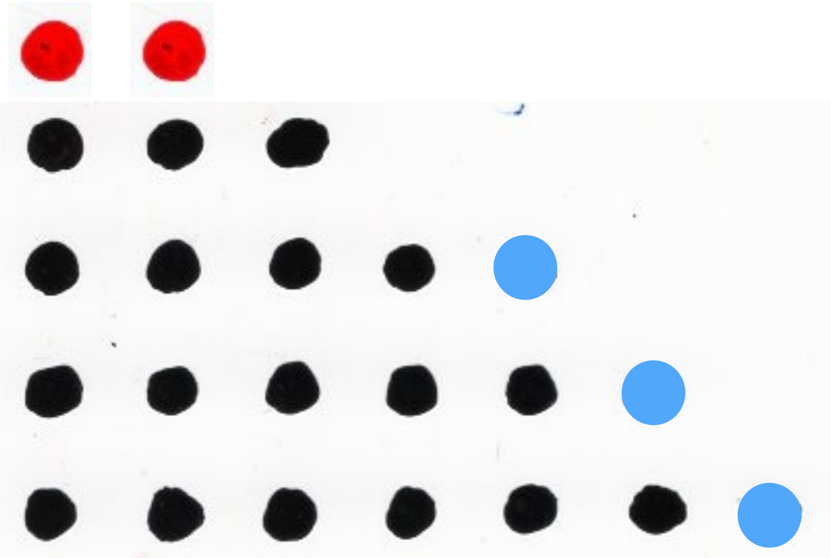


(ii) $r(F) > d(F)$

involution φ

$r(F)$ = number of cells of the top row of F

$d(F)$ = longest sequence of cells in diagonal position on the NE border of F

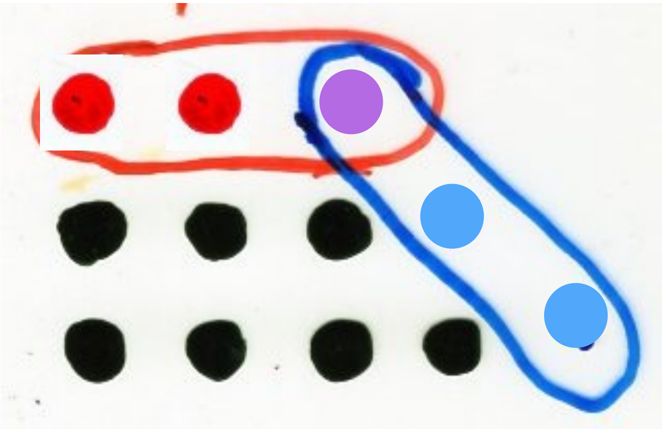


(i) $r(F) \leq d(F)$

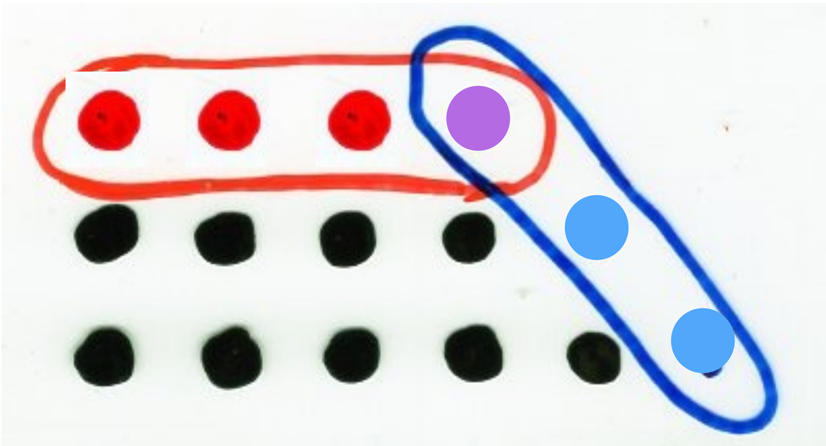
involution φ

$r(F)$ = number of cells of the top row of F

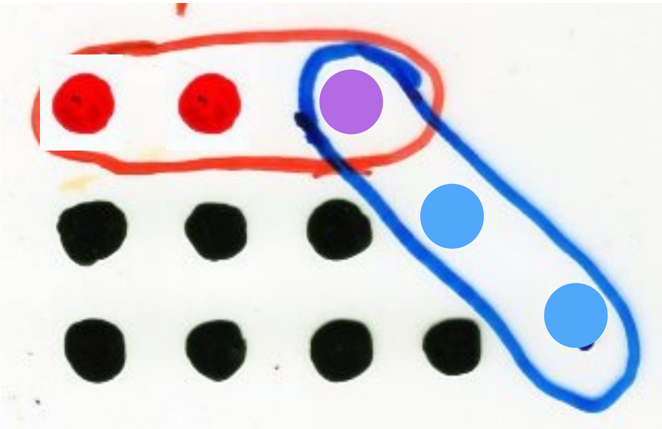
$d(F)$ = longest sequence of cells in diagonal position on the NE border of F

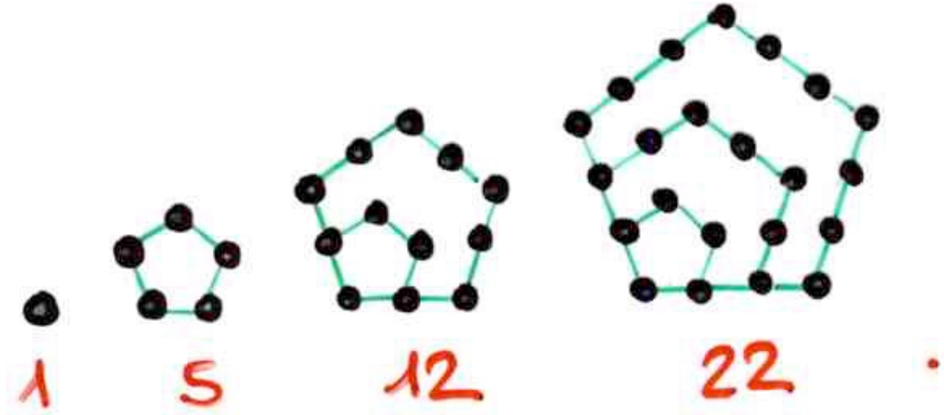


case where φ
is not defined



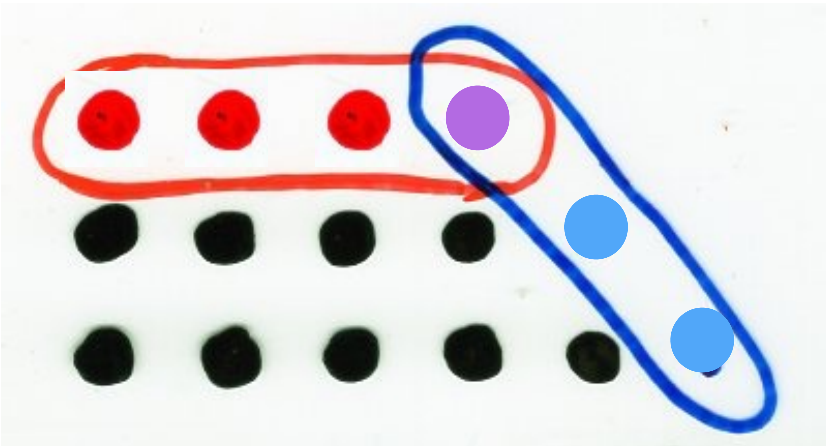
case where φ
is not defined





$$q^{n(3n-1)/2}$$

pentagonal numbers



$$q^{n(3n+1)/2}$$

$$\prod_{i\geq 1}(1-q^i) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

$$= \sum_{n\geq 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right)$$

Euler pentagonal identity

Franklin (1881)
bijection

sign-reversing
involution

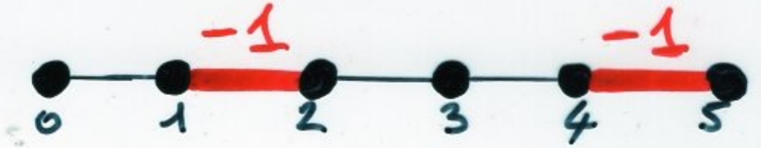
another exercise
with a sign-reversing involution

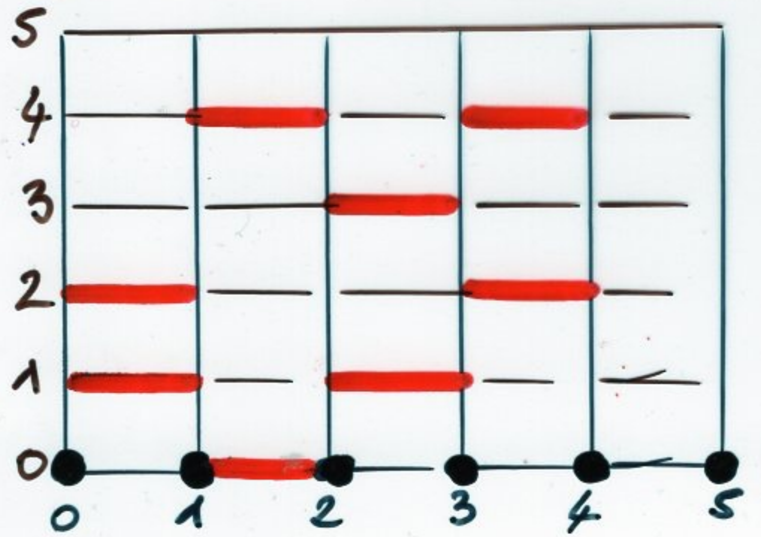
exercise

With the construction of a sign-reversing involution prove that the generating function for heaps of dimers on the segment $[0,k]$ is

$$\frac{1}{F_{k+1}(t)}$$

(Fibonacci polynomial)





(enumerated by the number of dimers)

more about rational series

Proposition $\mathbb{K}=\mathbb{C}$, $\beta_1, \dots, \beta_k \in \mathbb{C}$, $k \geq 1$, $\beta_k \neq 0$,
$\{a_n\}_{n\geq 0}$ with $a_n \in \mathbb{C}$.

The 3 following conditions are equivalent:

(i) $$\sum_{n\geq 0} a_n t^n = \frac{N(t)}{D(t)}$$ with $D = 1 + \beta_1 t + \dots + \beta_k t^k$
and N polynomial, degree $< k$

(ii) for every $n \geq 0$, $a_{n+k} + \beta_1 a_{n+k-1} + \dots + \beta_k a_n = 0$

(iii) Let $D = \prod_{i=1}^{r} (1-\lambda_i t)^{k_i}$. For every $n \geq 0$,

$$a_n = \sum_{i=1}^{r} P_i(n) \lambda_i^n,$$ with P_i polynomial
degree $< k_i$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \qquad (n \geq 0)$$

Transition matrix A $\qquad \displaystyle\sum_{n \geq 0} a_n t^n = \frac{N_{ij}}{\det(I_n - At)}$

a_n expression with the inverse of the zeros of the polynomial $\det(I_n - At)$

= the eigenvalues of A

zeros of $F_n(x)$ Fibonacci

$$\sin((n+1)\theta) = \sin U_n(\cos\theta)$$

$$U_n(x) = M_n(2x) \qquad M_n^*(x) = x^n M_n(1/x) = F_n(x^2)$$

zeros of $U_n(x)$: $\left\{ \cos\left(\frac{k}{(n+1)}\pi\right) \quad k=1,\dots,n \right\}$

inverse of zeros of $F_n(x)$: $\left\{ 4\cos^2\left(\frac{k}{n+1}\pi\right), k=1,\dots,n \right\}$

$$\frac{1}{F_{k+1}(t)}$$

$$\frac{F_k(t)}{F_{k+1}(t)}$$ bounded Dyck paths

$$T_n(x) = \frac{1}{2} C_n(2x) \qquad C_n^* = L_n(x^2) \qquad \cos(n\theta) = T_n(\cos\theta)$$

zeros of $T_n(x)$: $\left\{ \cos\left(\frac{(2k-1)}{2n} \pi \right) , \quad k = 1, \dots, n \right\}$

inverse of zeros of $L_n(x)$
Lucas polynomial

$\left\{ 4 \cos^2\left(\frac{(2k-1)}{2n} \pi \right) , \quad k = 1, \dots, n \right\}$

Lagrange inversion formula

Lagrange inversion formula

$\mathbb{K}$ field characteristic 0

$f(t)$ has a reciprocal power series
$y = f^{\langle -1 \rangle}(t)$ iff $f(0)=0$ and $f'(0) \neq 0$

let $f(t) = \dfrac{t}{\varphi(t)}$ with $\varphi(0) \neq 0$

y is the unique solution of
$y = t\,\varphi(y)$ (with $y(0)=0$)

Proposition (Lagrange inversion formula)

Let $g(t) \in \mathbb{K}[[t]]$, the coefficient of t^n in $g(y)$ is

$$[t^n]\, g(y) = \frac{1}{n} [t^{n-1}] \left(g'(t) \varphi(t)^n \right)$$

- analytic proof with residu of meromorph function
- combinatorial proof in Ch3

<u>example</u> Catalan generating function

$$y = \sum_{n \geq 1} C_n t^n, \quad y = 1 + t y^2, \quad z = 1 + y$$

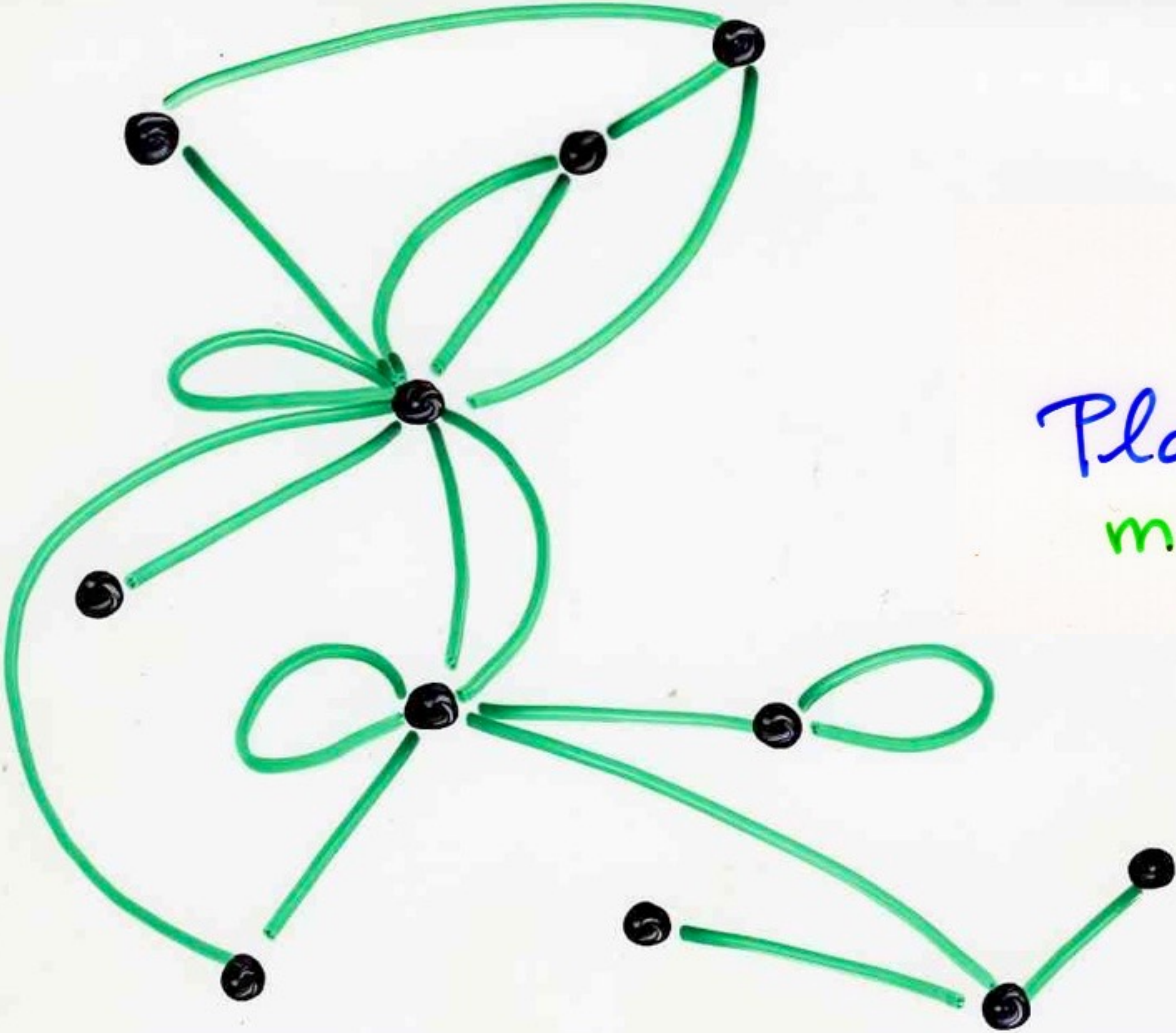
then $z = t\varphi(z)$ with $\varphi(t) = (1+t)^2$.

We have $C_n = \frac{1}{n} [t^{n-1}] (1+t)^{2n} = \frac{1}{n} \binom{2n}{n-1}$

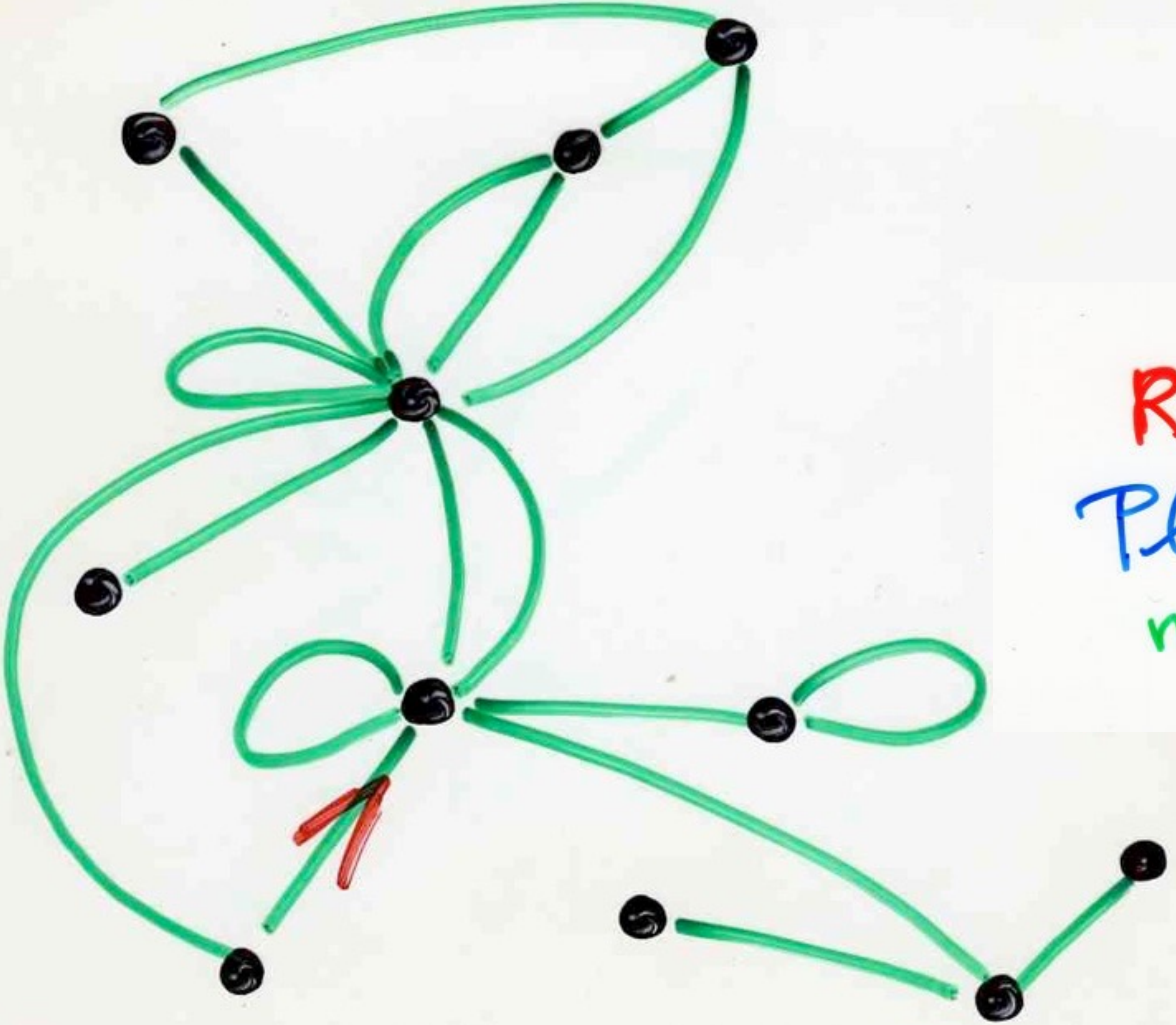
$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

algebraicity
with hidden decomposable structures

example: planar maps



Planar
map



Rooted
Planar
map

Tutte (1960)

a_n number of rooted planar maps with n edges

$$y = \sum_{n \geq 0} a_n t^n$$

$$y = \frac{1-4z}{(1-3z)^2}$$

$$z = \frac{t}{(1-3z)}$$

$$y = \frac{1-4z}{(1-3z)^2}$$

$$y = \sum_{n \geqslant 0} a_n t^n$$

$$z = \frac{t}{(1-3z)}$$

$$z = t + 3z^2$$

$$z = t\,h(t)$$

$$h = 1 + 3t h^2$$

$$3^n C_n$$

$$\begin{aligned} y &= (1 - 4t h)\, h^2 \\ &= h\,(h - 4t h^2) \\ &= h - t h^3 \end{aligned}$$

$$\begin{cases} h = 1 + 3t h^2 \\ y = h - t h^3 \end{cases}$$

Tutte (1960)

a_n number of rooted planar maps with n edges

$$y = \sum_{n \geq 0} a_n t^n$$

$$\begin{cases} h = 1 + 3t h^2 \\ y = h - t h^3 \end{cases}$$

$3^n C_n$

Catalan number

Cori, Vauquelin (1970)

Schützenberger

→ methodology
(below)

$$\begin{cases} h = 1 + 3t h^2 \\ y = h - t h^3 \end{cases}$$

coding planar maps
with words
in the difference
of two algebraic languages

Cori, Vauquelin (1970)

Lagrange inversion formula

$$y = \sum_{n \geqslant 0} a_n t^n$$

$$y = \frac{1-4z}{(1-3z)^2}$$

$$z = \frac{t}{(1-3z)}$$

$$\begin{cases} z = t\,\varphi(z) \\ y = g(z) \end{cases}$$

with $$\begin{cases} \varphi(t) = \dfrac{1}{1-3t} \\ g(t) = \dfrac{1-4t}{(1-3t)^2} \end{cases}$$

$$
\begin{aligned}
a_n &= \frac{1}{n}\,[t^{n-1}]\left(g'(t)\,(1-3t)^{-n}\right)\\
&= \frac{2}{n}\,[t^{n-1}]\left((1-6t)(1-3t)^{-n-3}\right)\\
&= \frac{2}{n}\,[t^{n-1}]\,(1-3t)^{-n-3} - 6\,[t^{n-2}](1-3t)^{-n-3}\\
&= \frac{2}{n}\left(3^{n-1}\times\frac{(n+3)\cdots(2n+1)}{(n-1)!} - 6\times 3^{n-2}\times\frac{(n+3)\cdots(2n)}{(n-2)!}\right)\\
&= \frac{2}{n}\,3^{n-1}\,\frac{(n+3)\cdots(2n)}{(n-2)!}\left(\frac{2n+1}{n-1}-2\right)\\
&= 2\times 3^{n}\,\frac{(n+3)\cdots(2n)}{n!}\\
&= 2\times 3^{n}\,\frac{(2n)!}{(n+2)!\,n!}\\
&= 2\times\frac{3^{n}}{(n+2)}\,C_n
\end{aligned}
$$

$$a_n = \frac{2\times 3^{n}}{(n+2)}\,C_n$$

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Schaeffer (1997)

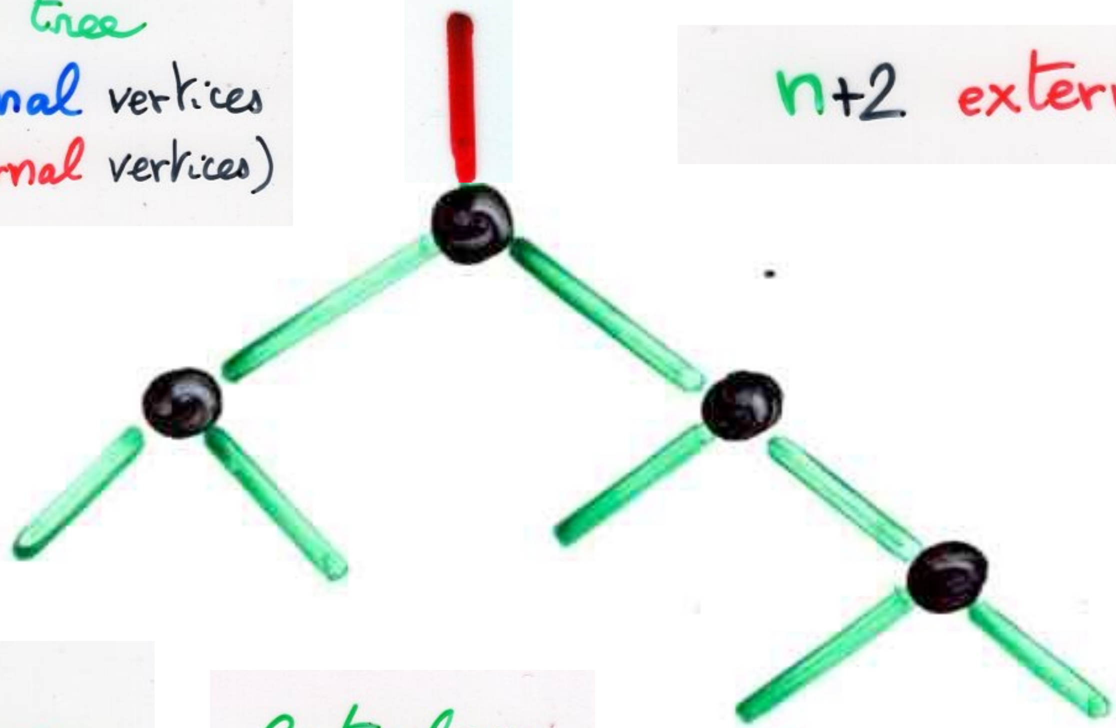
binary tree
with n internal vertices
(or $n+1$ external vertices)

$$\frac{2 \times 3^n}{(n+2)} C_n$$

Catalan number

Schaeffer (1997)

binary tree
with n internal vertices
(or $n+1$ external vertices)



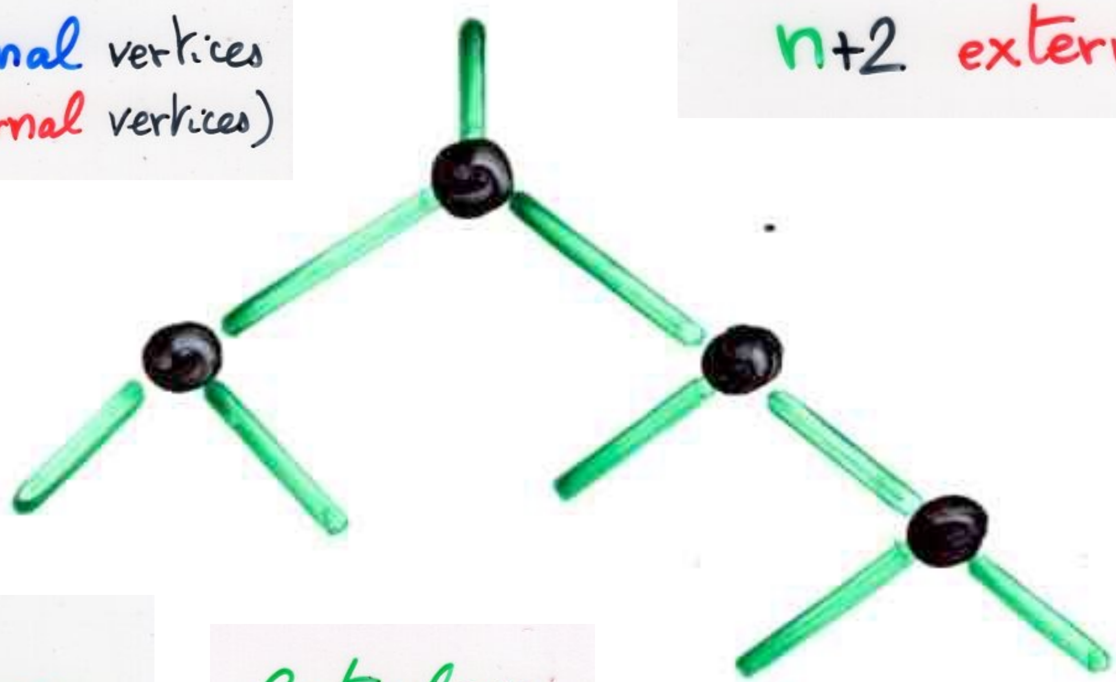
$n+2$ external edges

$$\frac{2 \times 3^n}{(n+2)} C_n$$

Catalan number

Schaeffer (1997)

binary tree
with n internal vertices
(or $n+1$ external vertices)



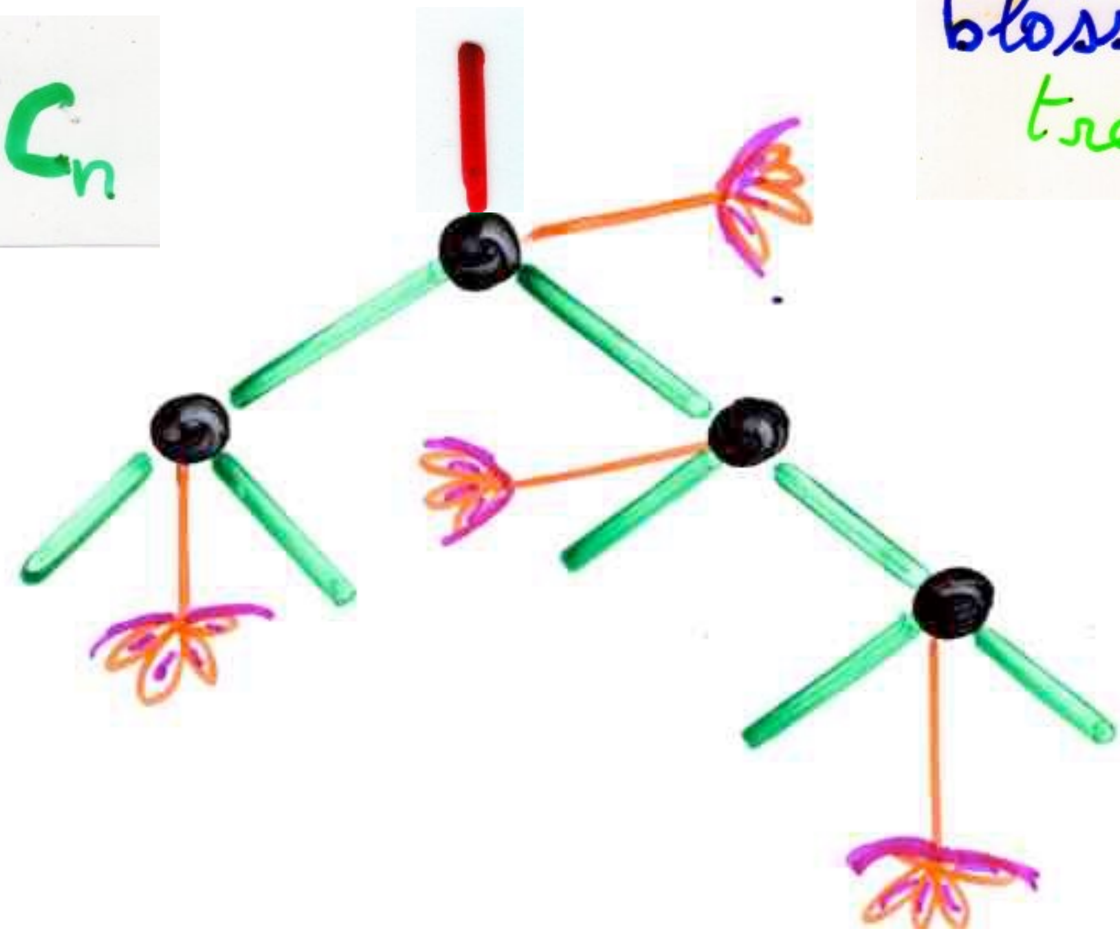
$n+2$ external edges

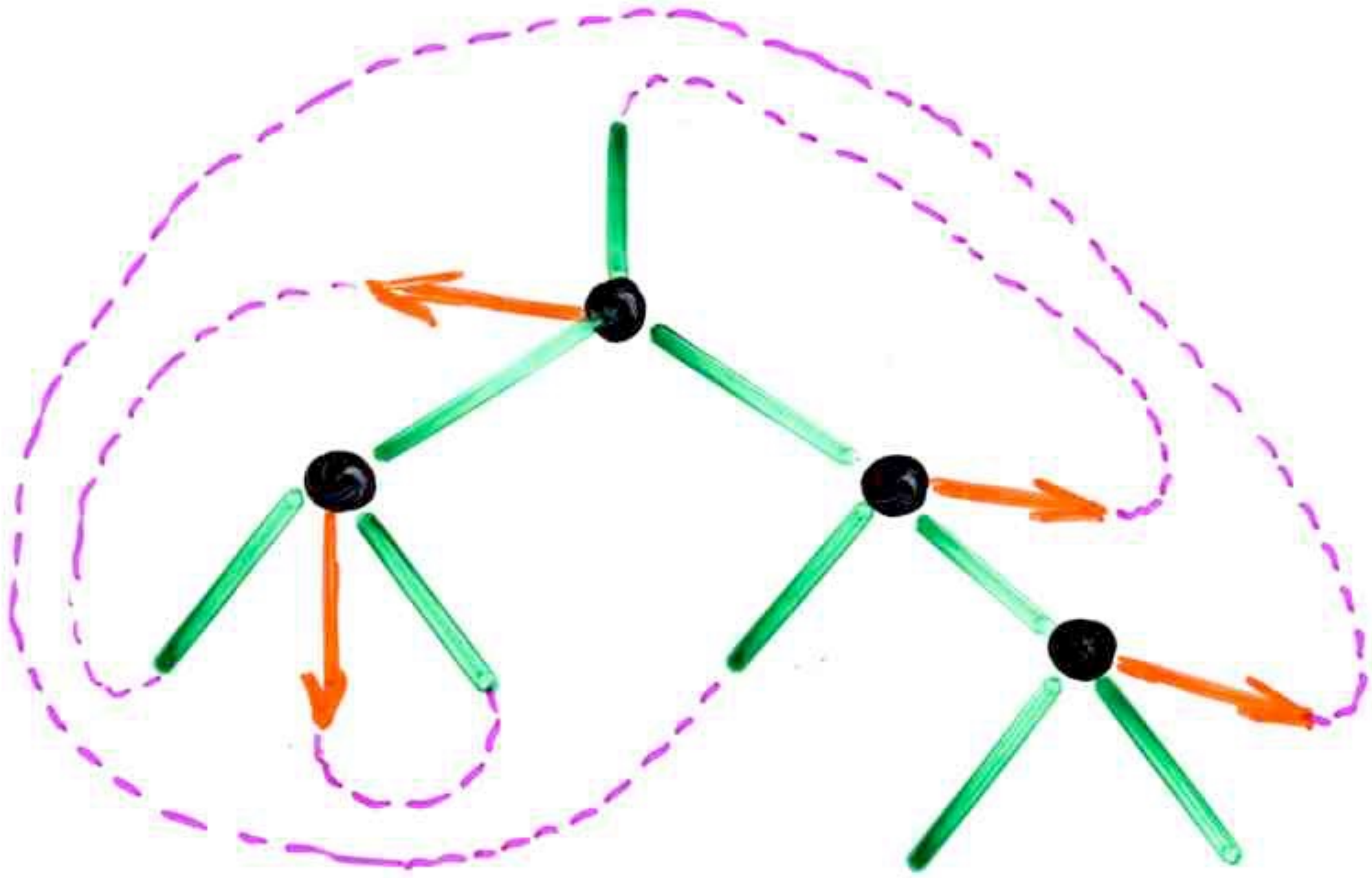
$$\frac{2 \times 3^n}{(n+2)} C_n$$

Catalan number

$3^n C_n$

blossoming trees



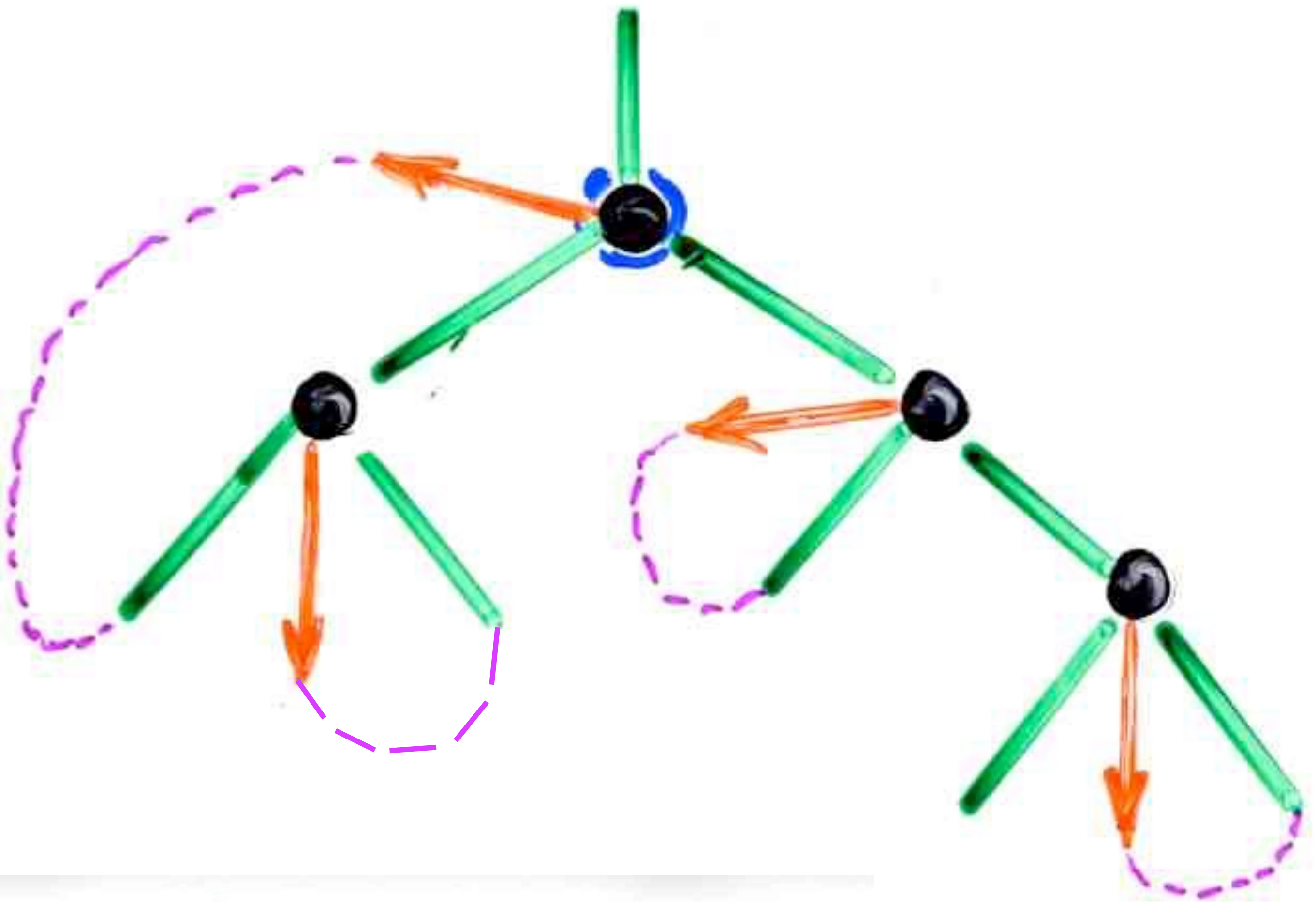


bijection
- planar maps (n edges)
- balanced blossoming trees (n nodes)

Schaeffer (1997)

not balanced

→ see chapter 2



bijection

- planar maps (n edges)
- balanced blossoming trees (n nodes)

Schaeffer (1997)

balanced

→ see chapter 2

Cori, Vauquelin (1970)

Schaeffer (1997)

Bouttier, Di Francesco, Guitter (2002)

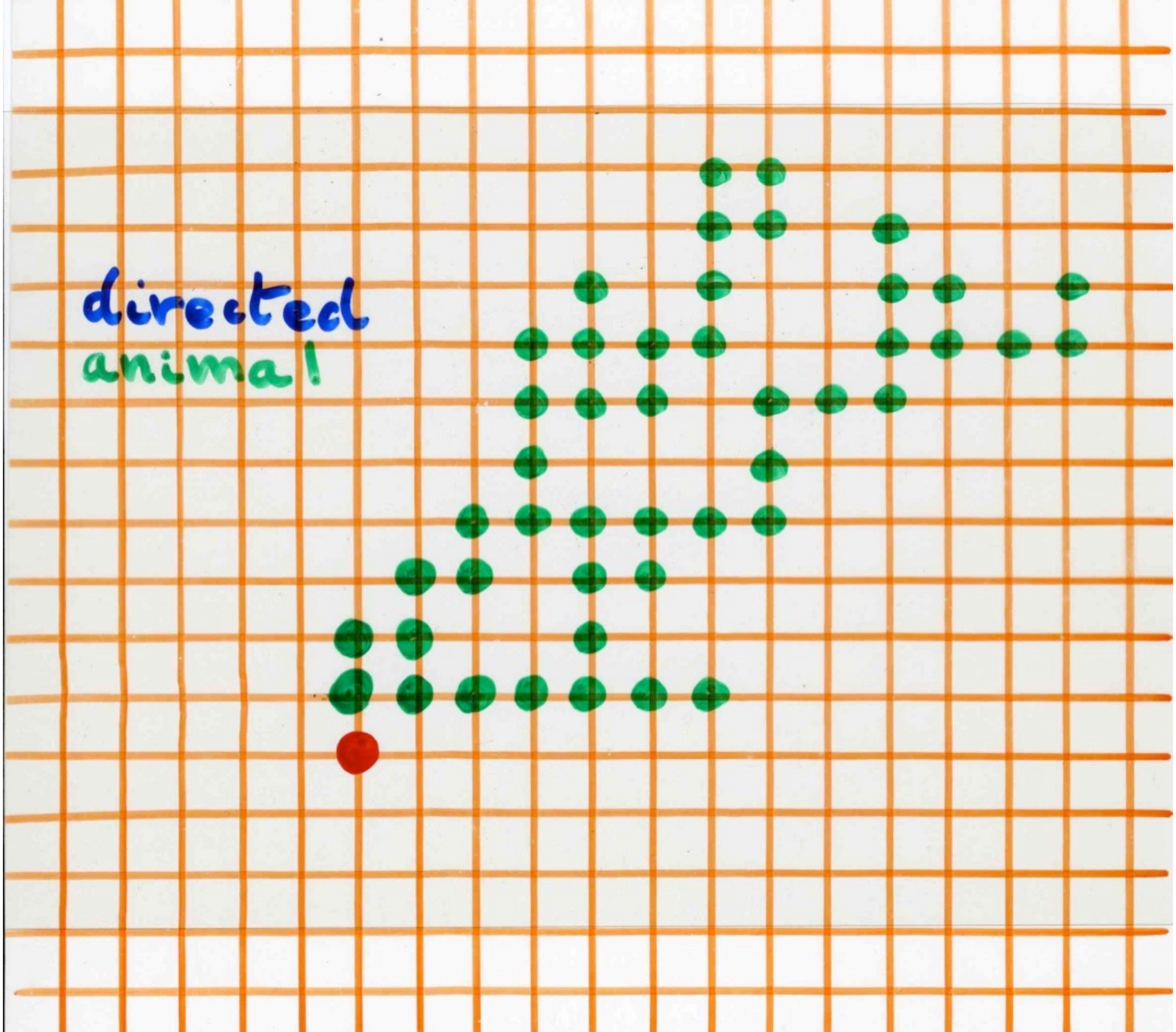
. . . many others

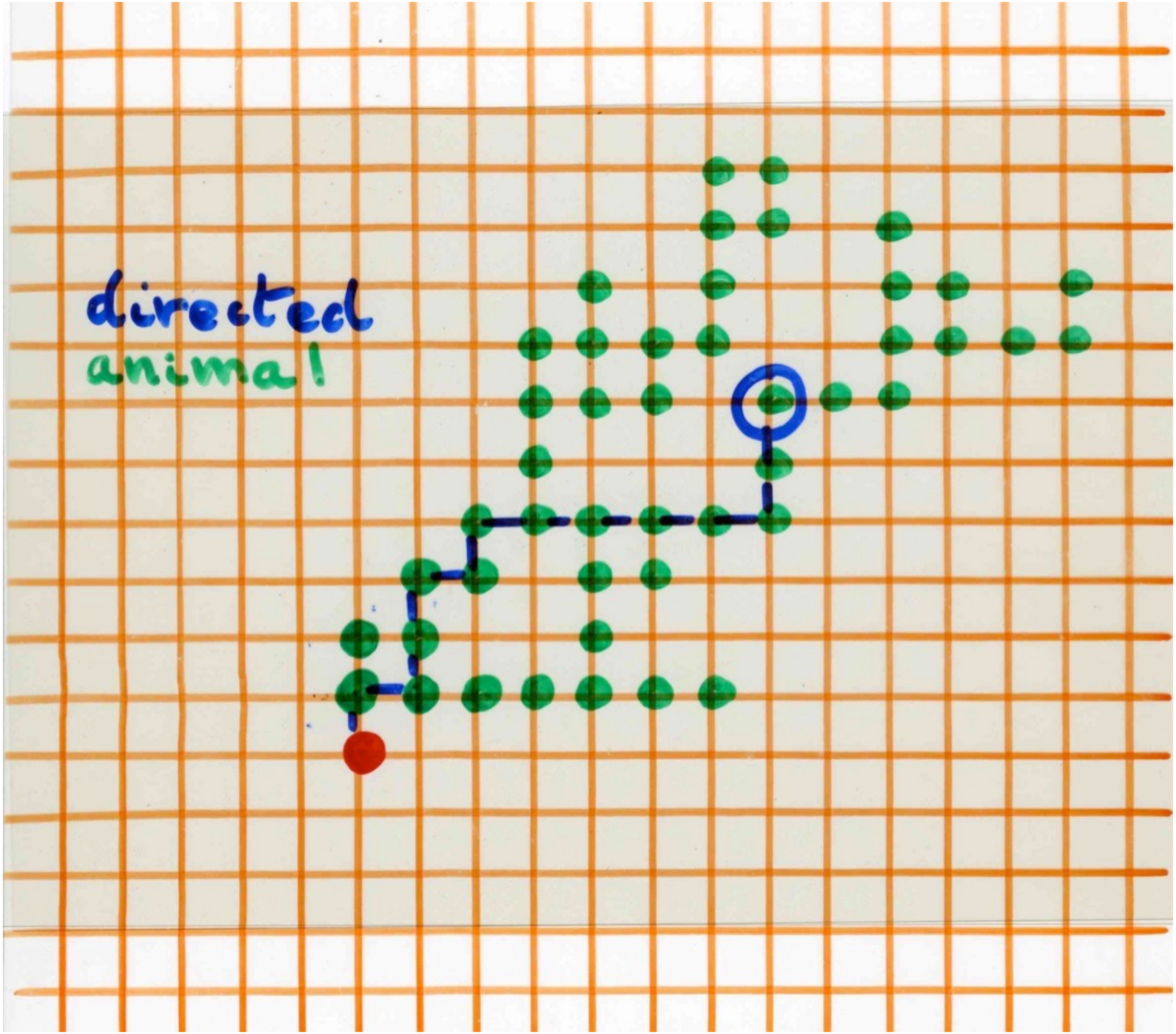
quantum
gravity

complements

algebraicity
with hidden decomposable structures

example: directed animals


directed
animal


directed
animal

y generating function
for the number of
directed animal
with n points
satisfies the system of
algebraic equations :

$$y = z + yz$$

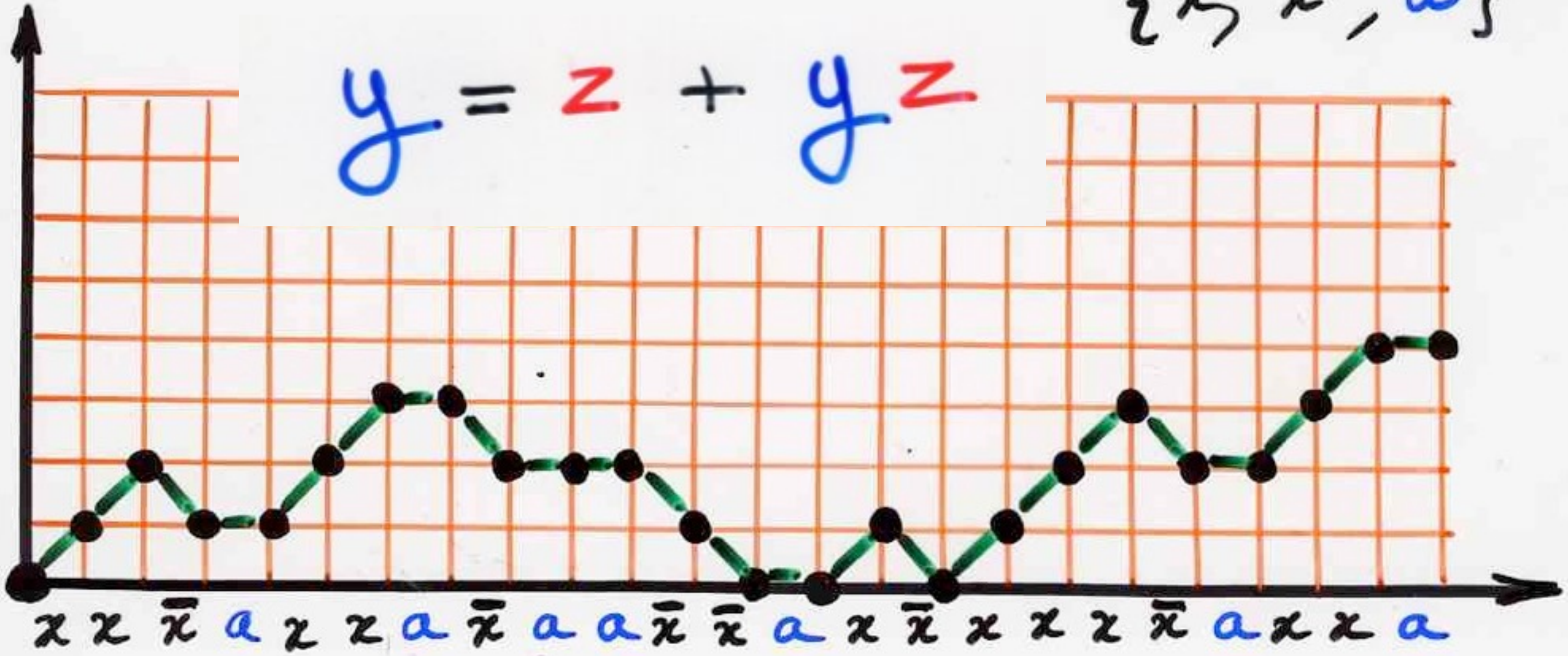
$$z = t + tz + tz^2$$

exercise algebraic equations for Motzkin paths and prefix of Motzkin paths

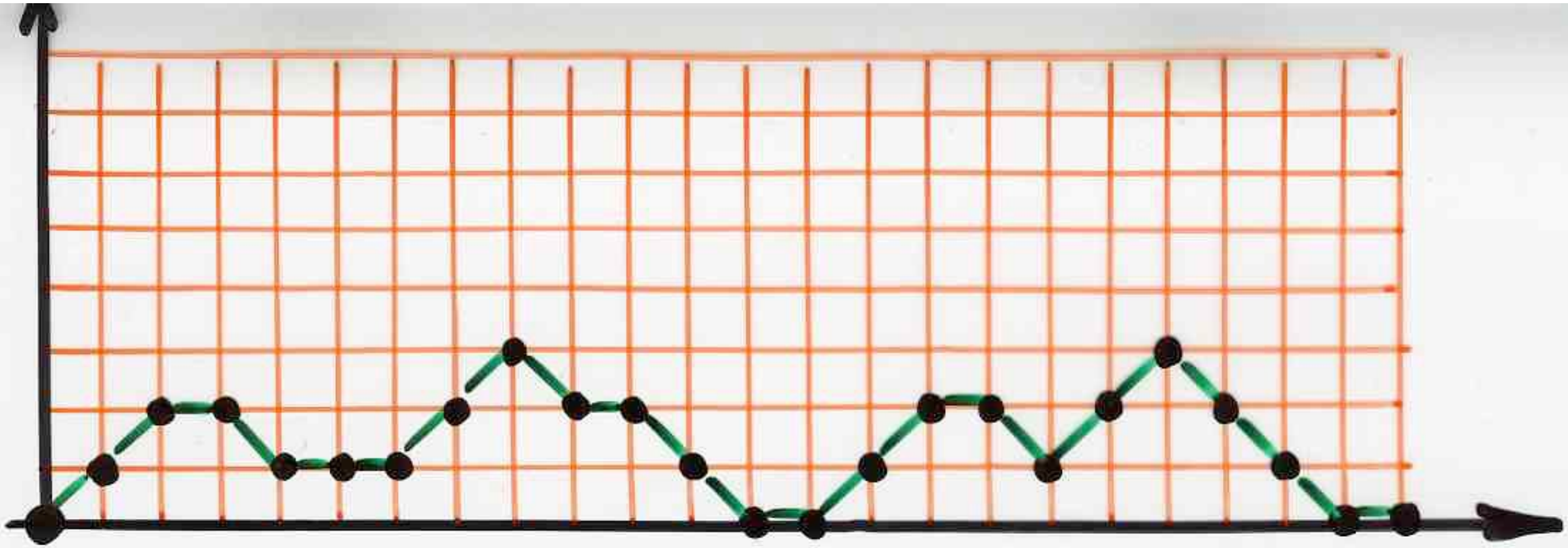
prefix (left factor) of a Motzkin path (word)

$\{x, \bar{x}, a\}$

$$y = z + yz$$



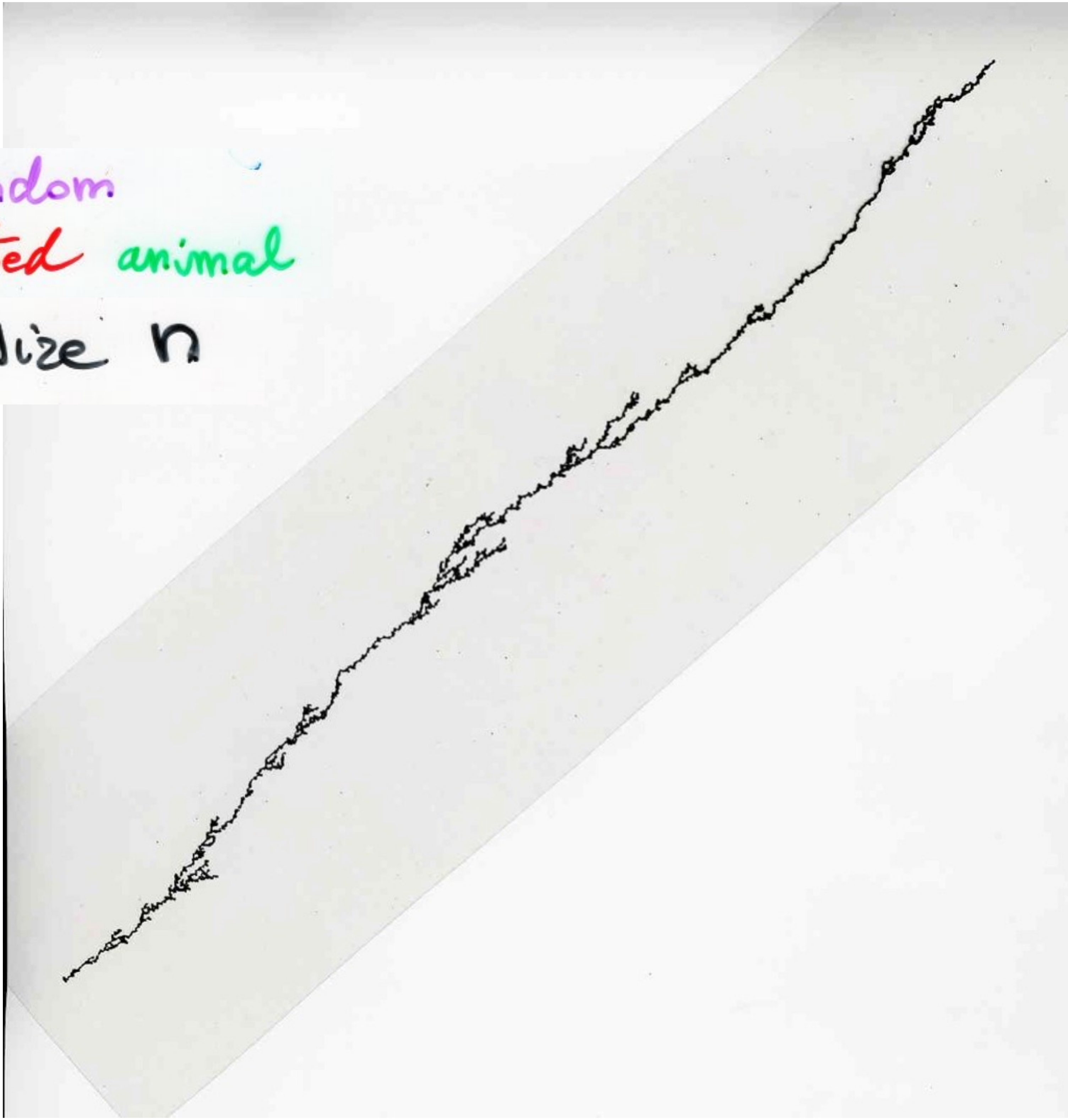
Motzkin path



$$z = t + tz + tz^2$$

random directed animal

size n



random
directed animal
size n

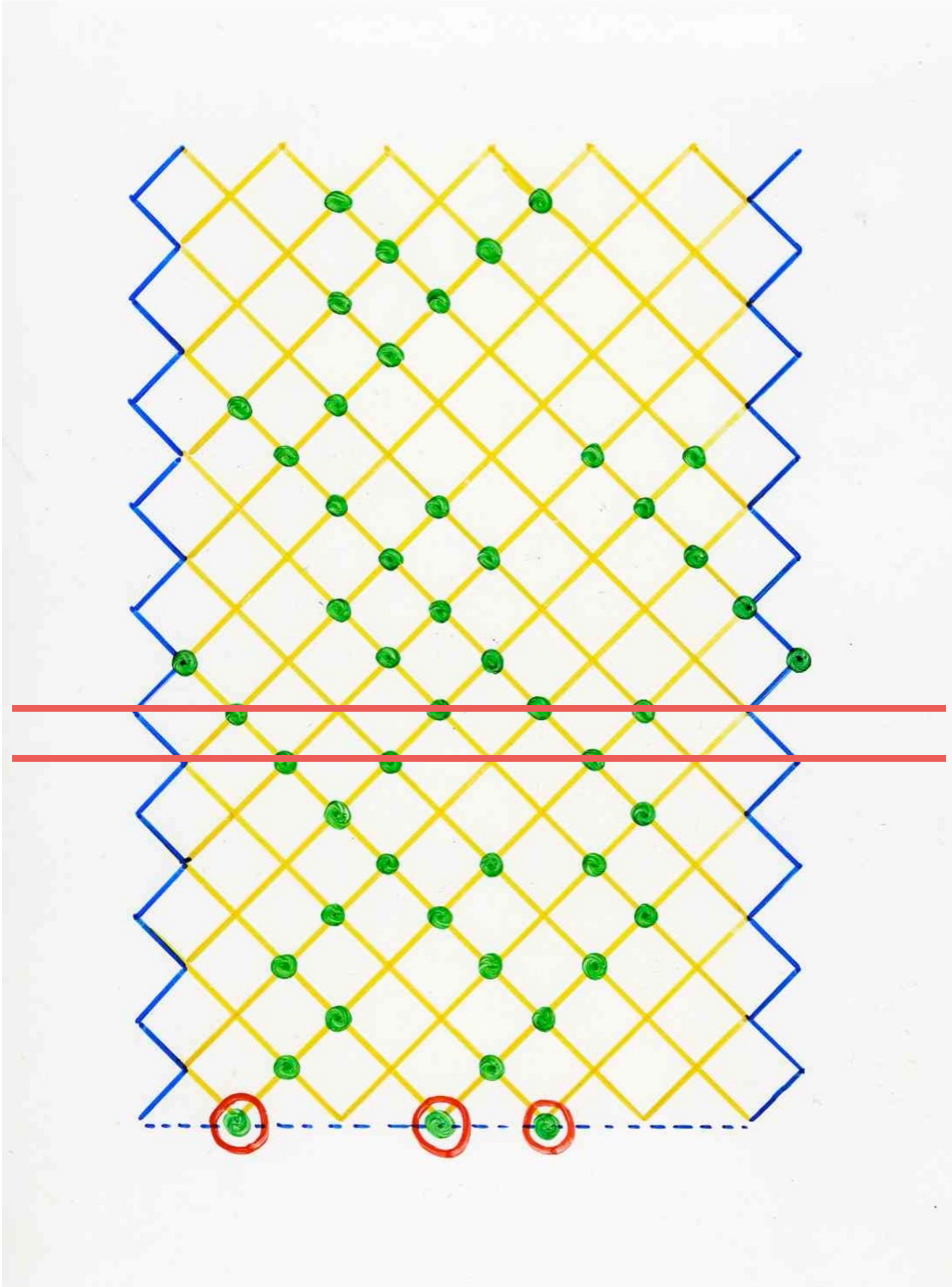
an anecdote .....

J. Vannimenus

J.P. Nadal

(1982, 1983)

B. Derrida



directed
animals
on a
circular
strip

$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin \alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin (i+\frac{1}{2})\, \alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} (1 + 2\cos \alpha_p)^{n-1}$$

animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k}\pi$$

B. Derrida

J. Vannimenus

J.P. Nadal

(1982, 1983)

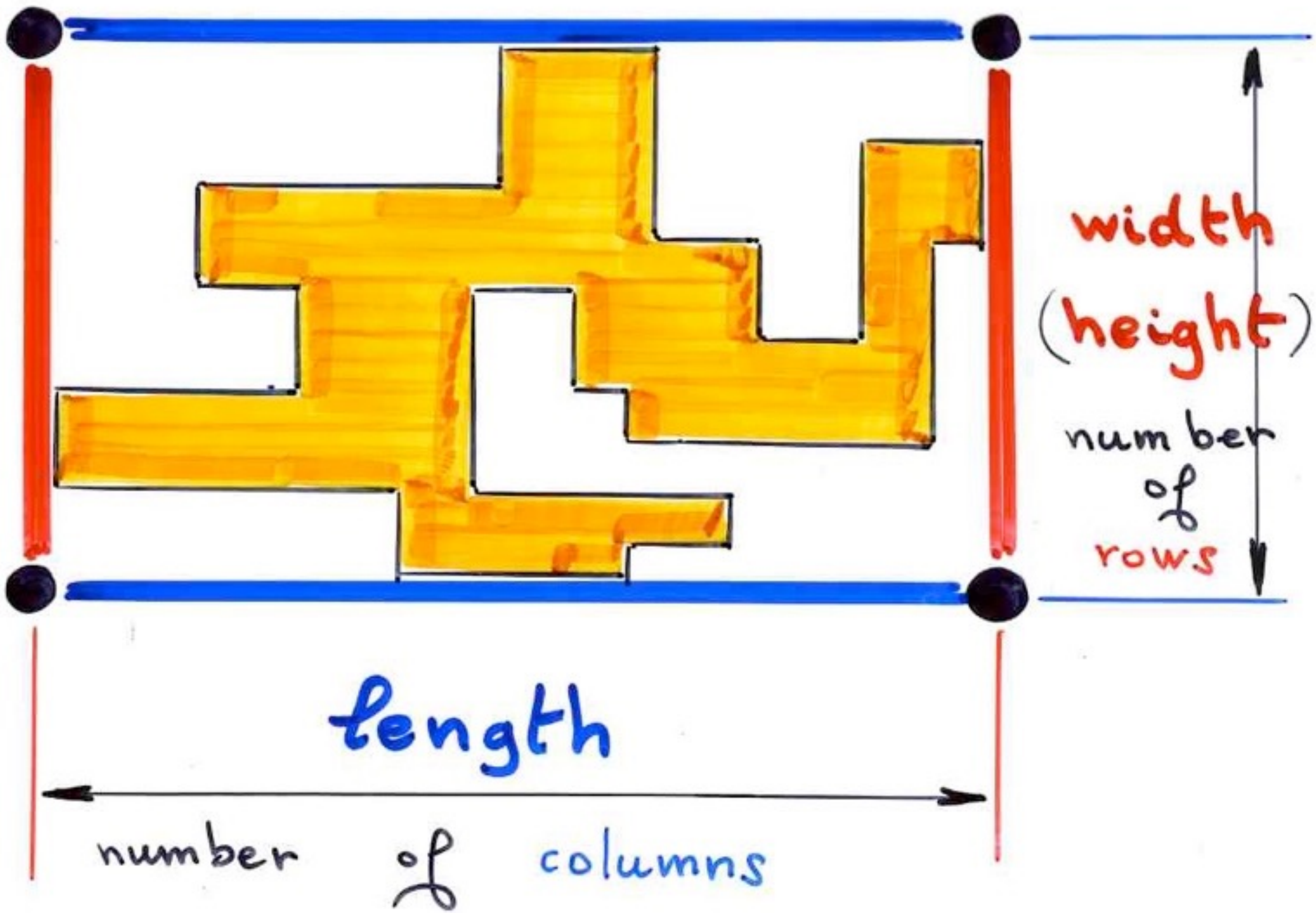
$$T_n(x) = \frac{1}{2} C_n(2x) \qquad C_n^* = L_n(x^2) \qquad \cos(n\theta) = T_n(\cos\theta)$$

zeros of $T_n(x)$: $\left\{ \cos\left(\frac{(2k-1)}{2n} \pi \right), \quad k = 1, \dots, n \right\}$

complements

algebraicity
with hidden decomposable structures

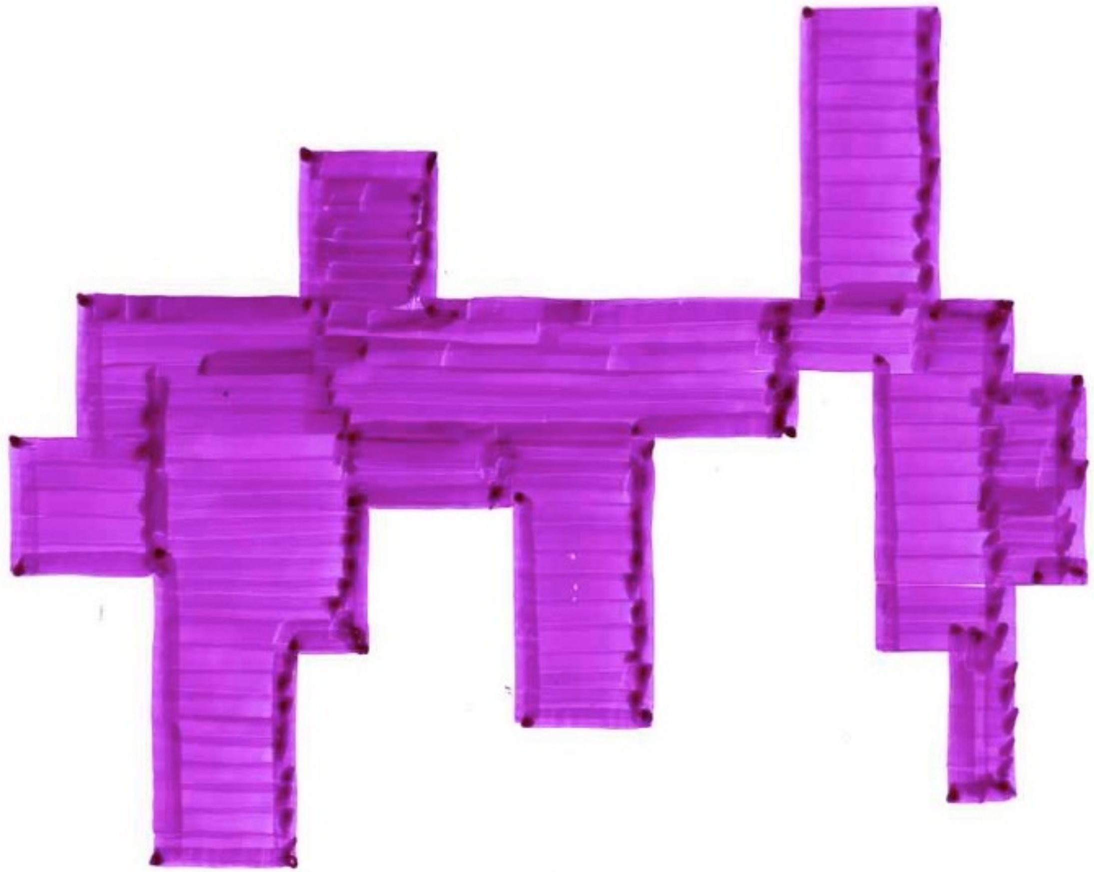
example: convex polyominoes

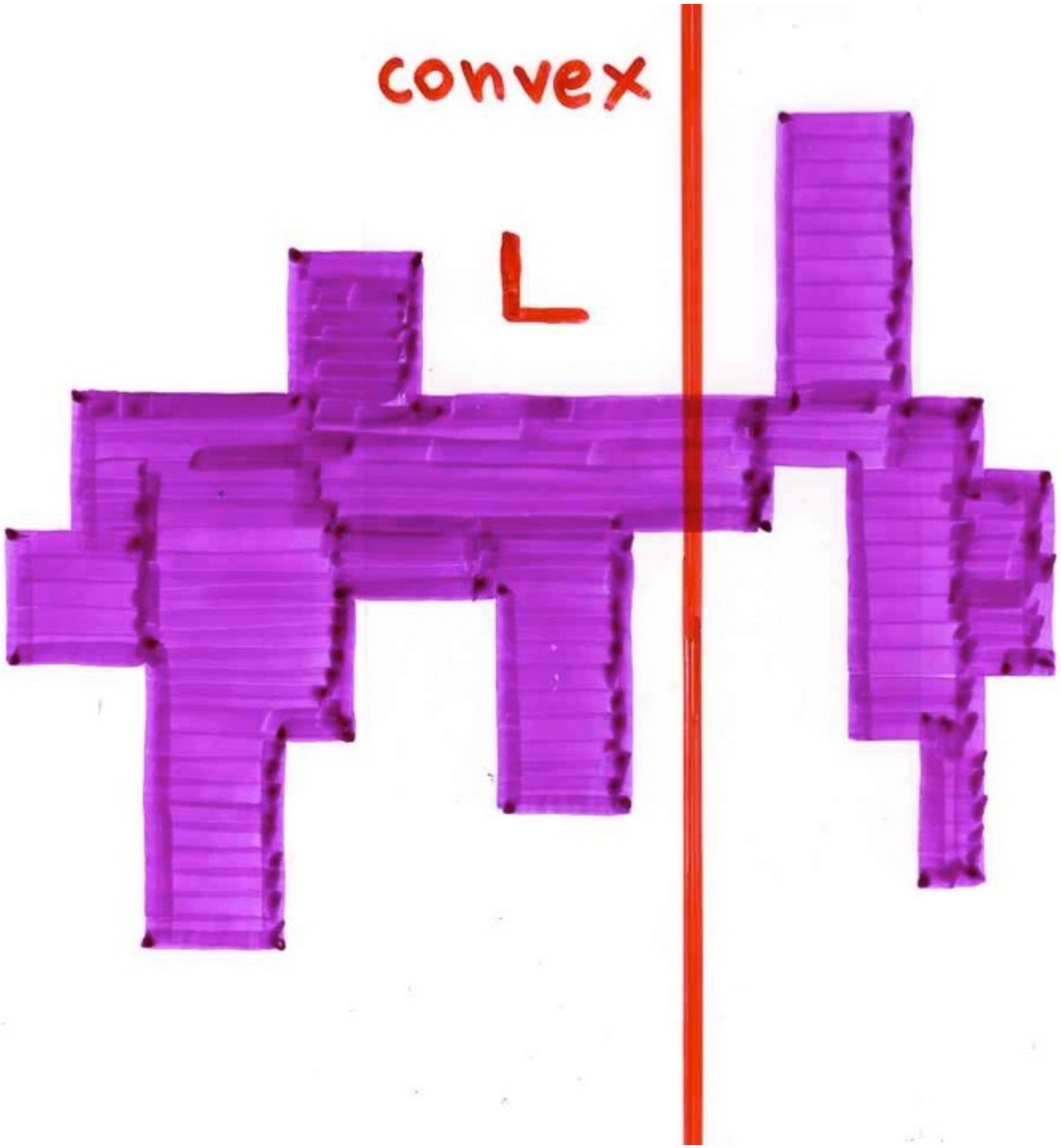

width
(height)
number
of
rows
length
number of columns

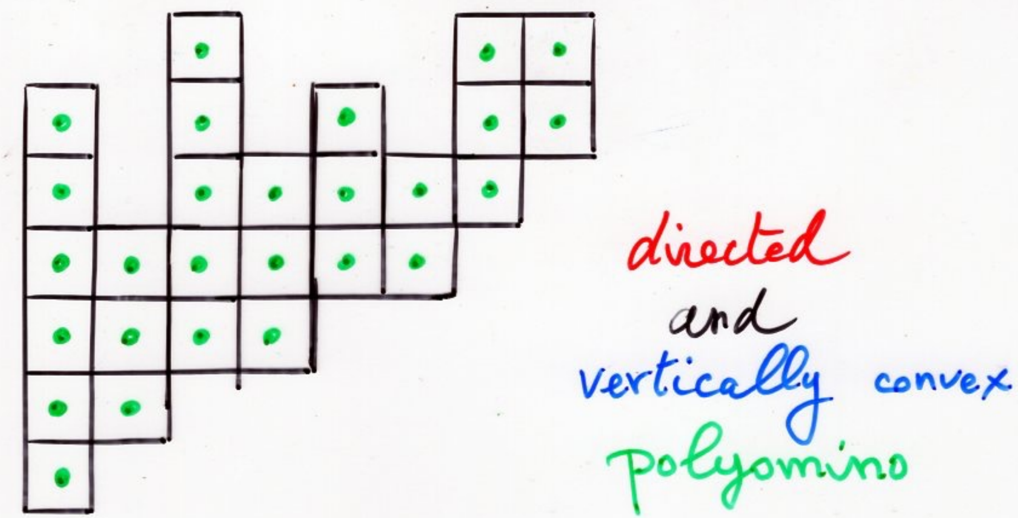
Polyominoes enumeration

a_n = number of polyominoes with **area** n

b_m = number of polyominoes with **perimeter** m




convex
L



a_n number of such polyominoes with n cells

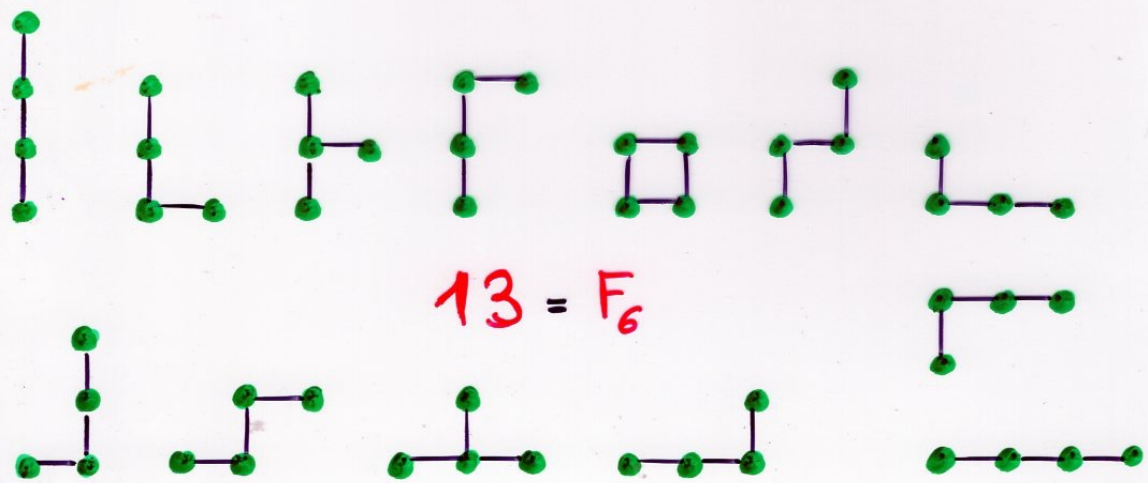
$a_n = F_{2n-2}$ Fibonacci numbers

$a_n = F_{2n-2}$ Fibonacci numbers

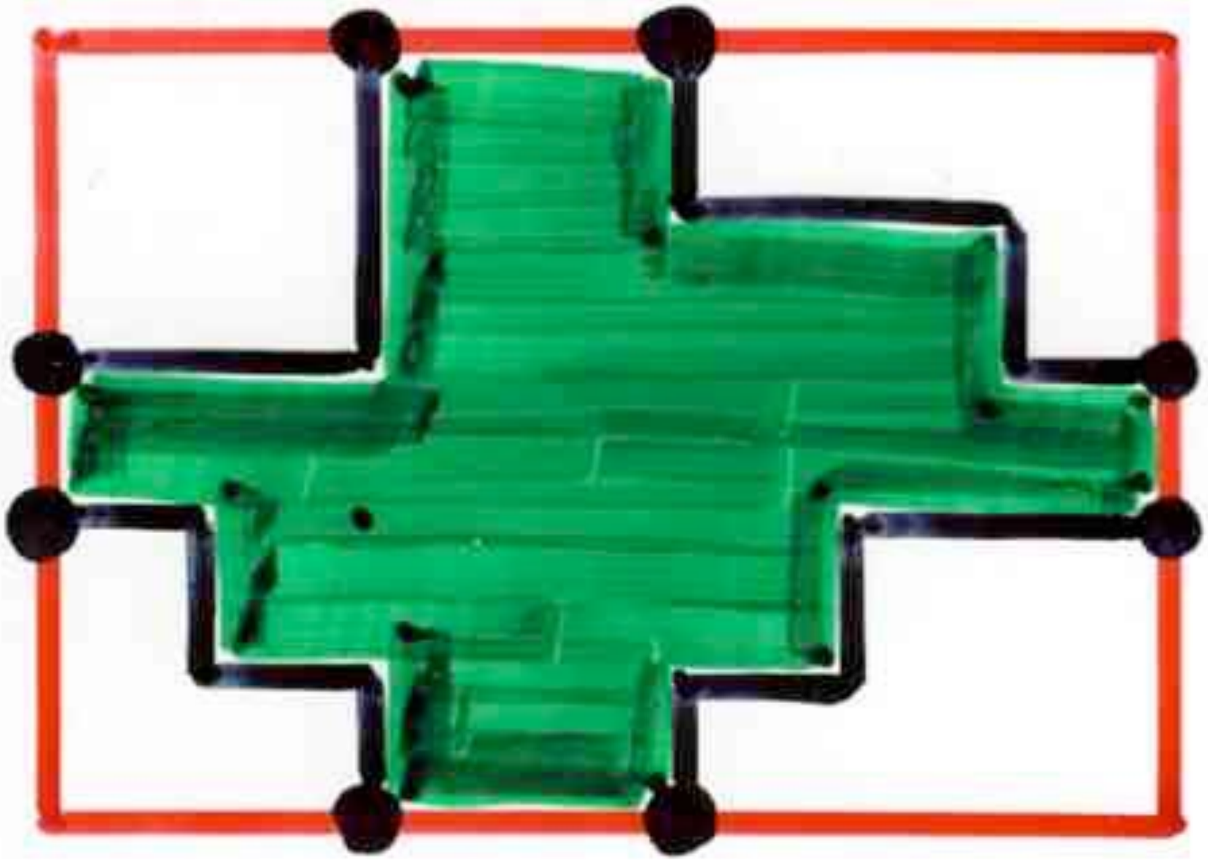
$1 = F_0$

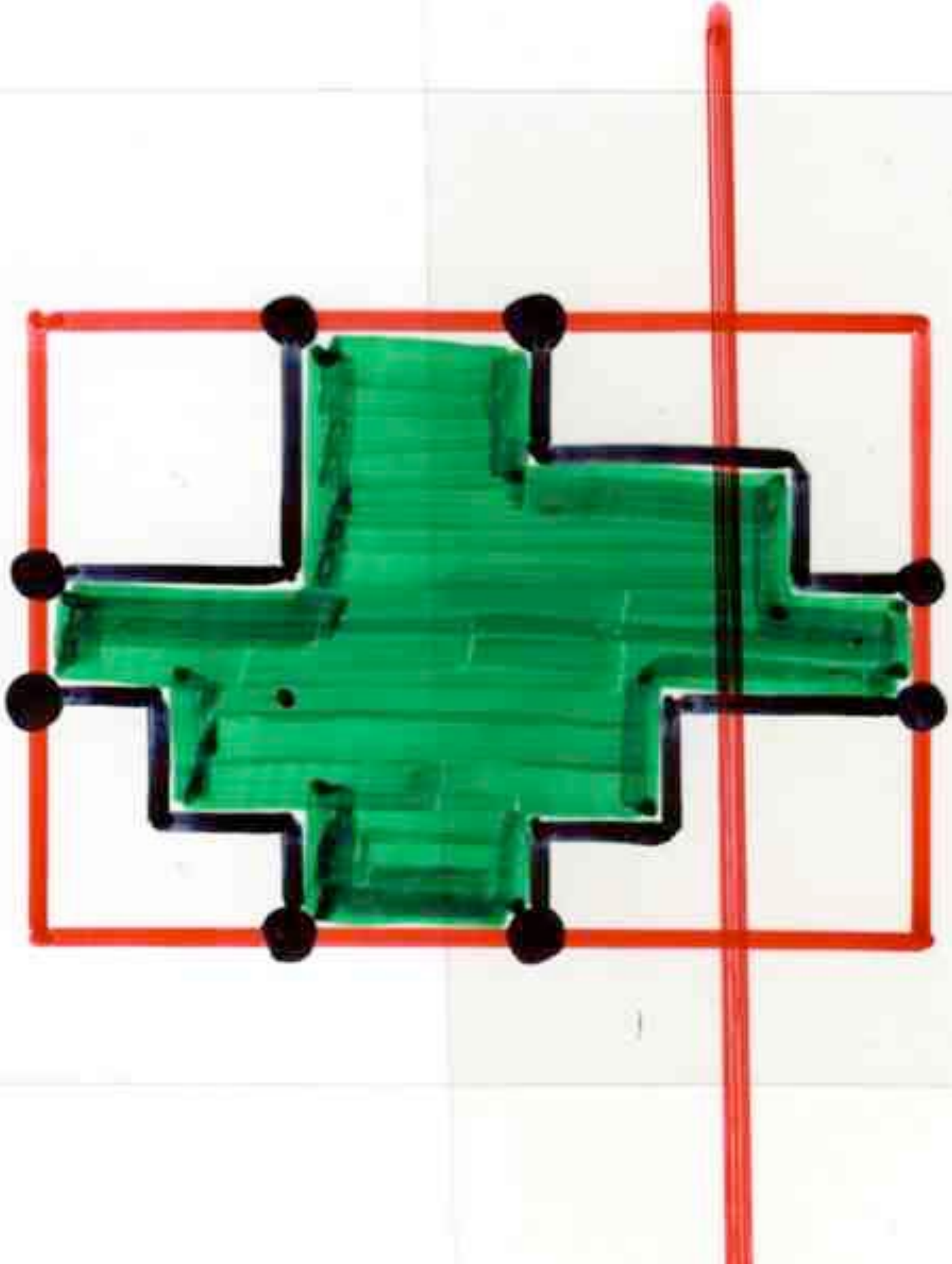
$2 = F_2$

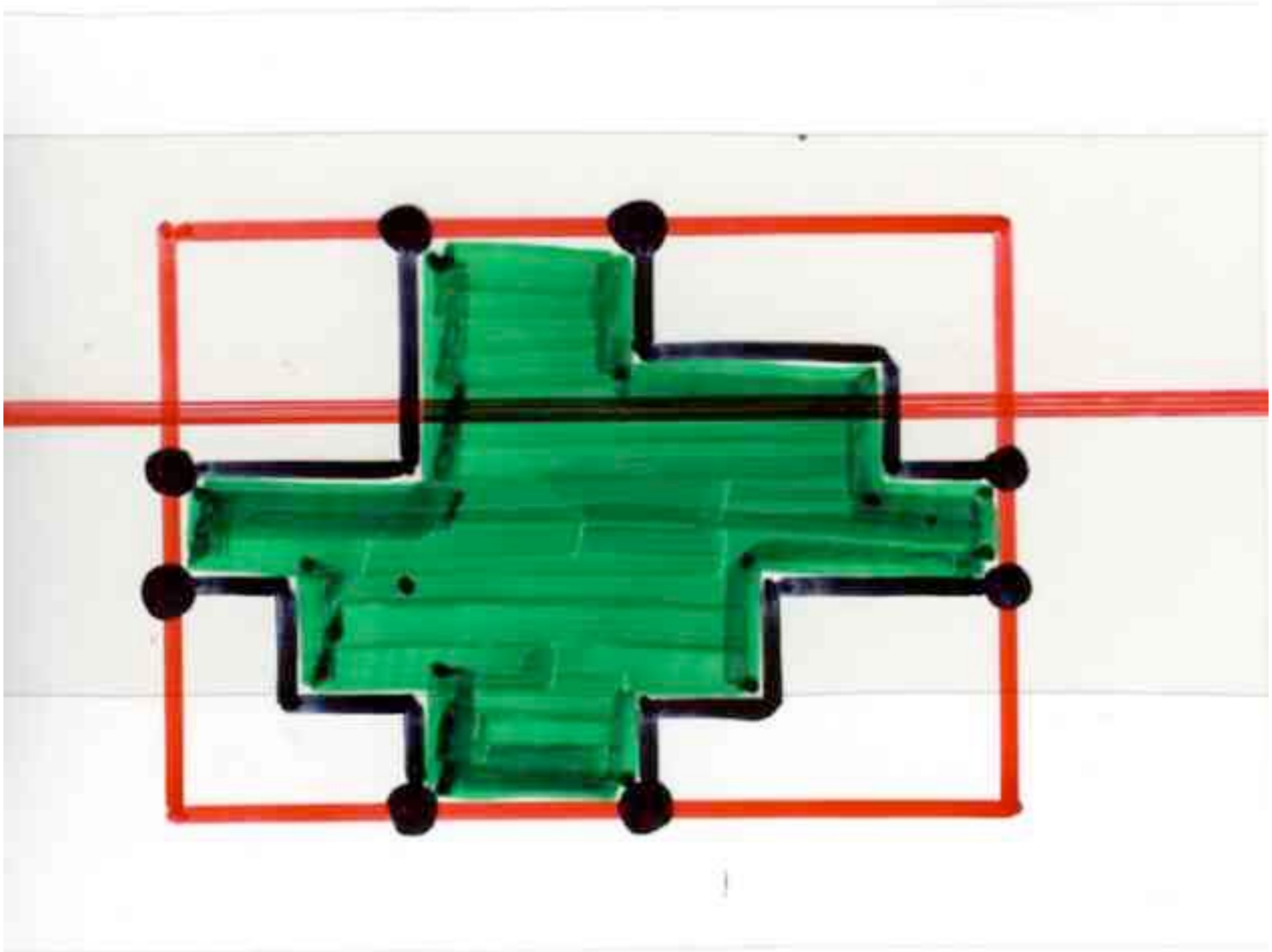
$5 = F_4$



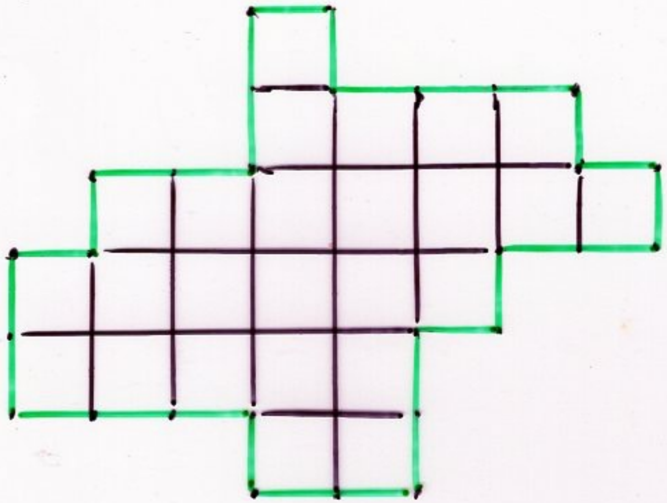
$13 = F_6$







convex polyominoes



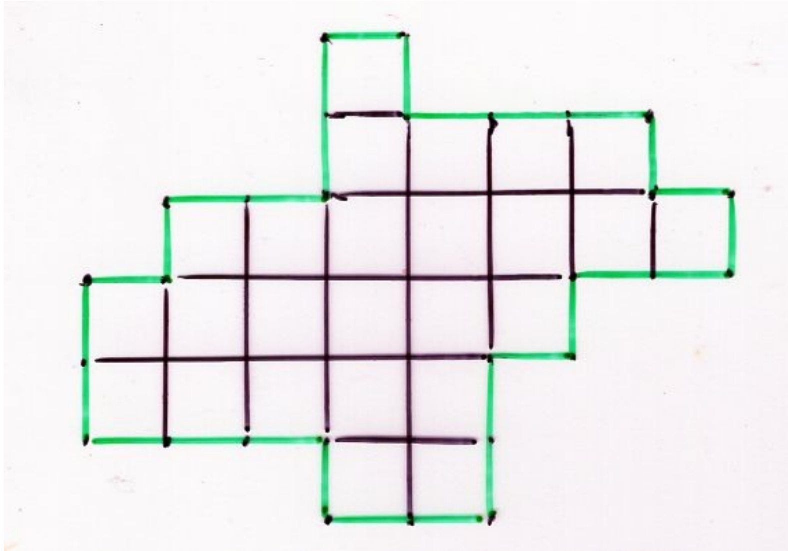
P_{2n} = number of convex polyominoes with perimeter $2n$

$$\sum_{n\geq 2} P_{2n} t^{2n} = \frac{t^4(1-6t^2+11t^4-4t^6)}{(1-4t^2)^2} - 4t^8(1-4t^2)^{-3/2}$$

Delest, X.V. (1984)

$$P_{2n+8} = (2n+11)4^n - 4(2n+1)\binom{2n}{n} \quad (n\geq 0)$$

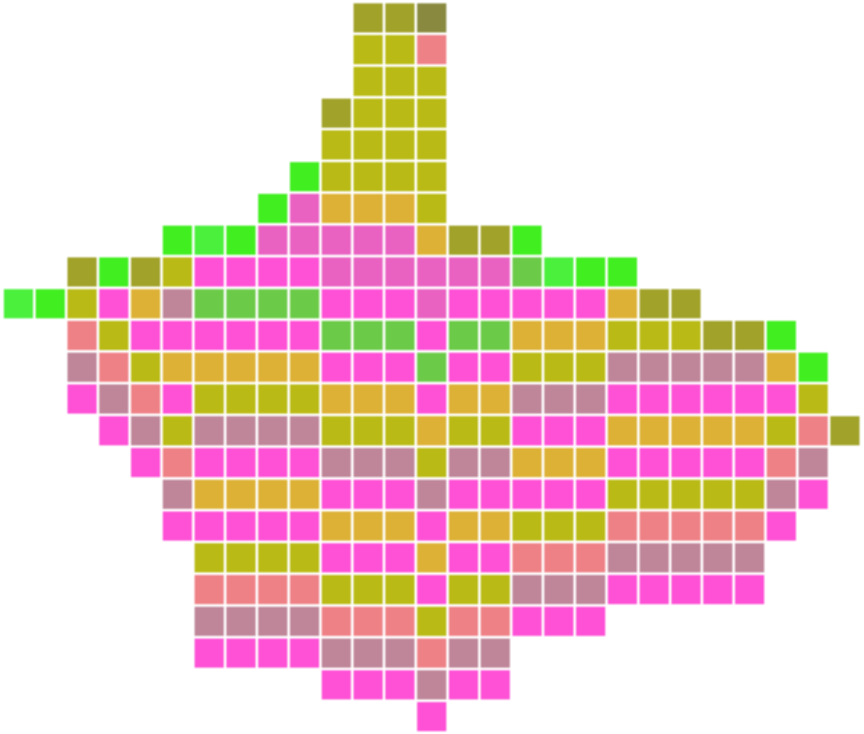
convex polyominoes



P_{2n} = number of convex polyominoes with perimeter $2n$

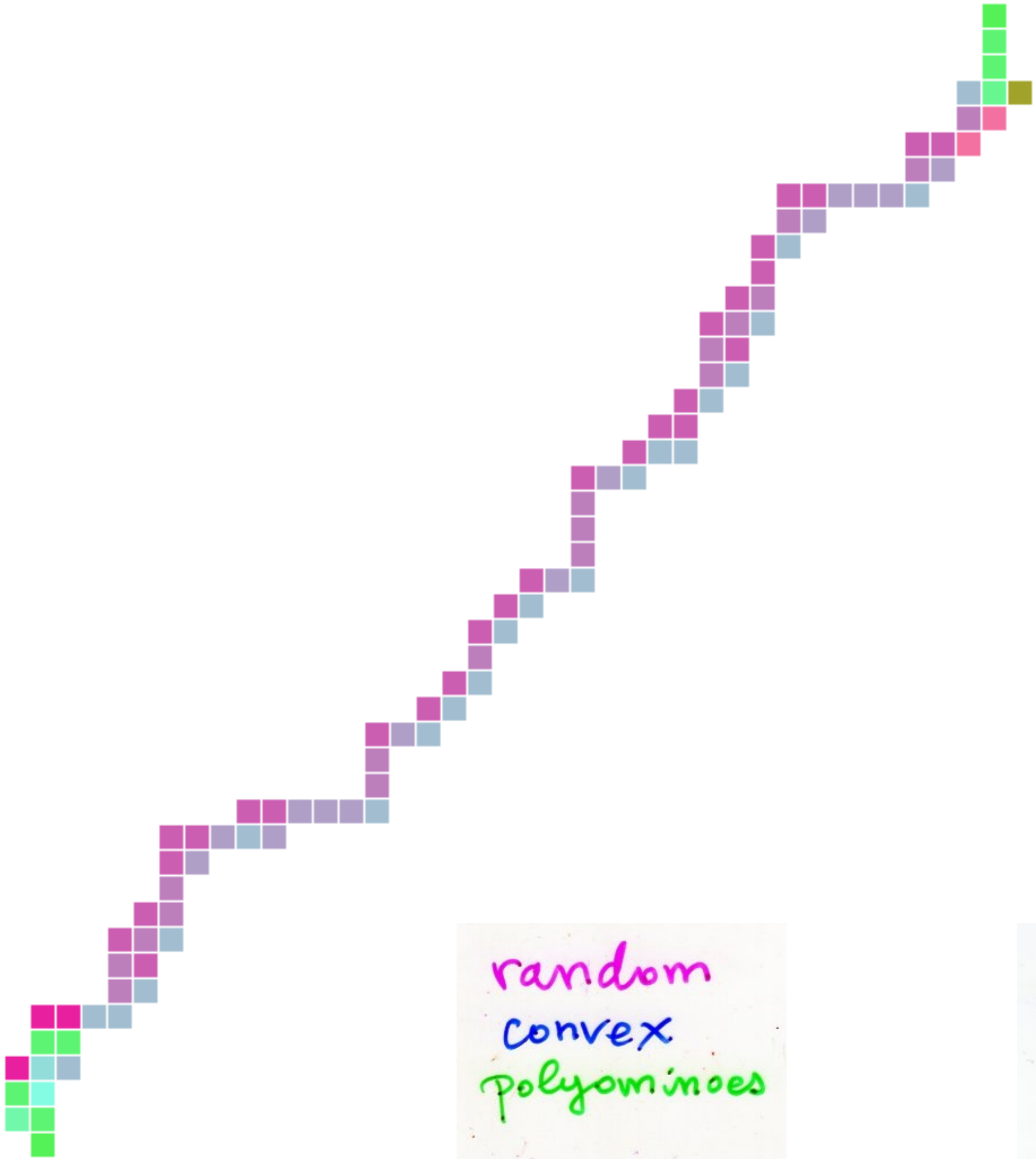
coding convex polyominoes with words of algebraic languages

→ Schützenberger methodology (below)



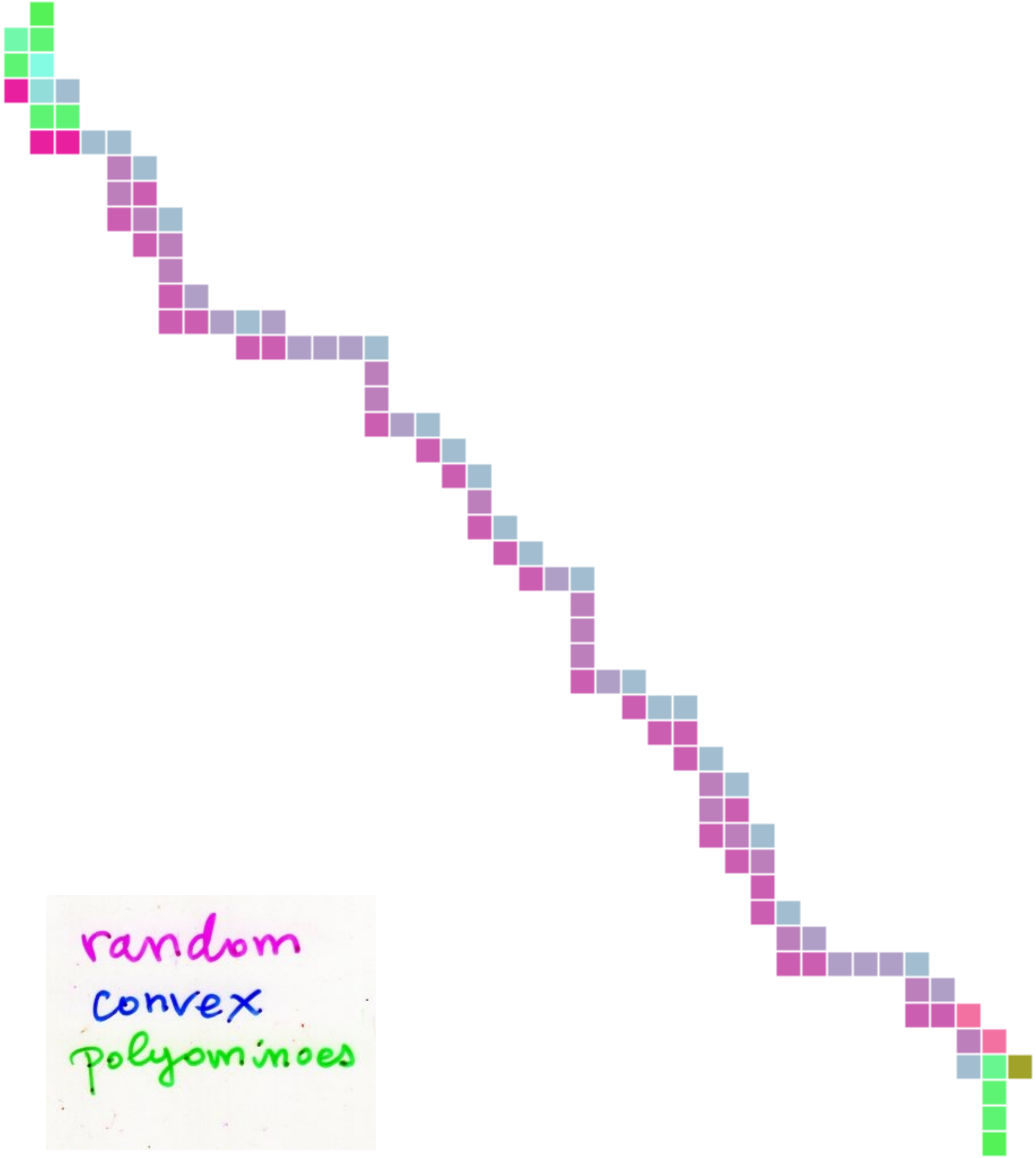
random
convex
polyominoes

with fixed
perimeter



random
convex
polyominoes

with fixed
area



random
convex
polyominoes

with fixed
area

rational and algebraic generating functions

A flavor of theoretical computer science
with

Schütengerger methodology
coding with words of algebraic languages

Schützenberger methodology

coding of combinatorial objects with words of algebraic language

context-free (algebraic) language

algebraic grammar

$$D \rightarrow TD \quad \text{or} \quad D \rightarrow \varepsilon \quad \text{(empty word)}$$

$$T \rightarrow x D \bar{x}$$

context-free (algebraic) language

algebraic grammar

$D \to TD$ or $D \to \varepsilon$ (empty word)

$T \to x D \bar{x}$

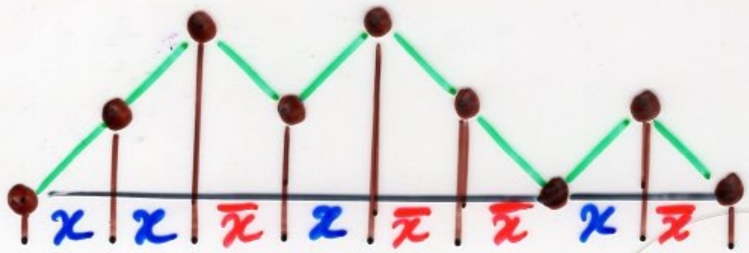
$$D$$
$$T\,D$$
$$x\,D\,\bar{x}\,D$$
$$x\,T\,D\,\bar{x}\,D$$
$$x\,x\,D\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,T\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,D\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,T\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,D\,\bar{x}$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}$$

(restricted) Dyck language $D \subseteq \{x, \bar{x}\}^*$

(i) $|w|_x = |w|_{\bar{x}}$ (nb of occurrences of x in the word w)

(ii) for every factorization $w = uv$ $\quad |u|_x \geq |u|_{\bar{x}}$

Coding of Dyck paths



C_n Catalan nb

only (i) : bilateral Dyck language

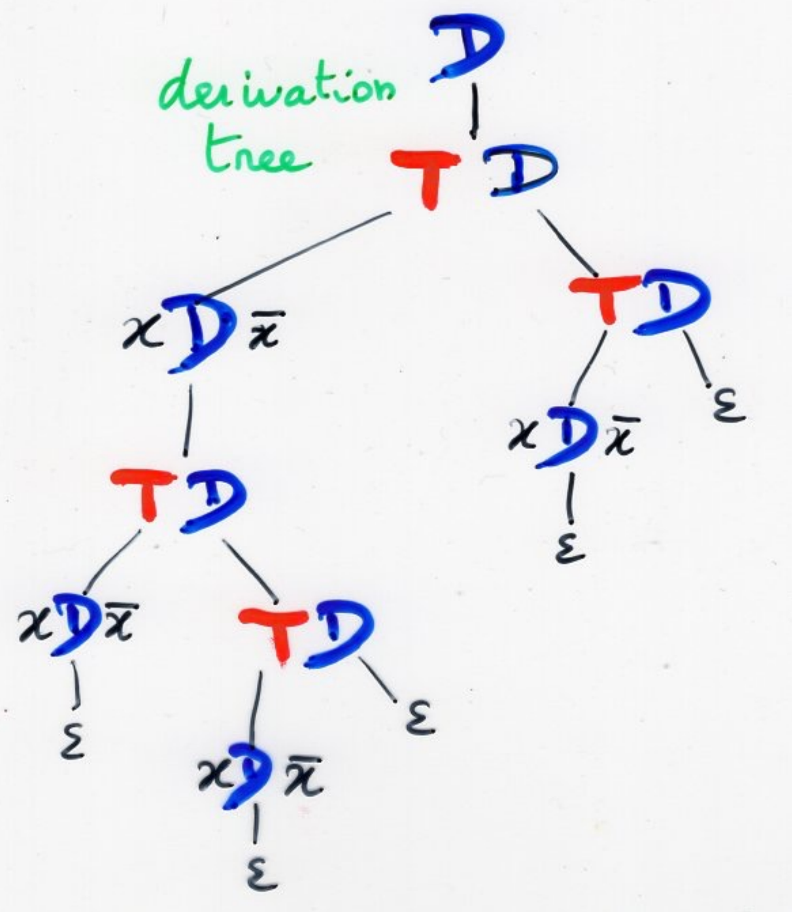
coding bilateral Dyck paths

enumerated by $\binom{2n}{n}$

algebraic grammar

$$D \to T D \quad \text{or} \quad D \to \varepsilon \quad \text{(empty word)}$$

$$T \to x D \bar{x}$$

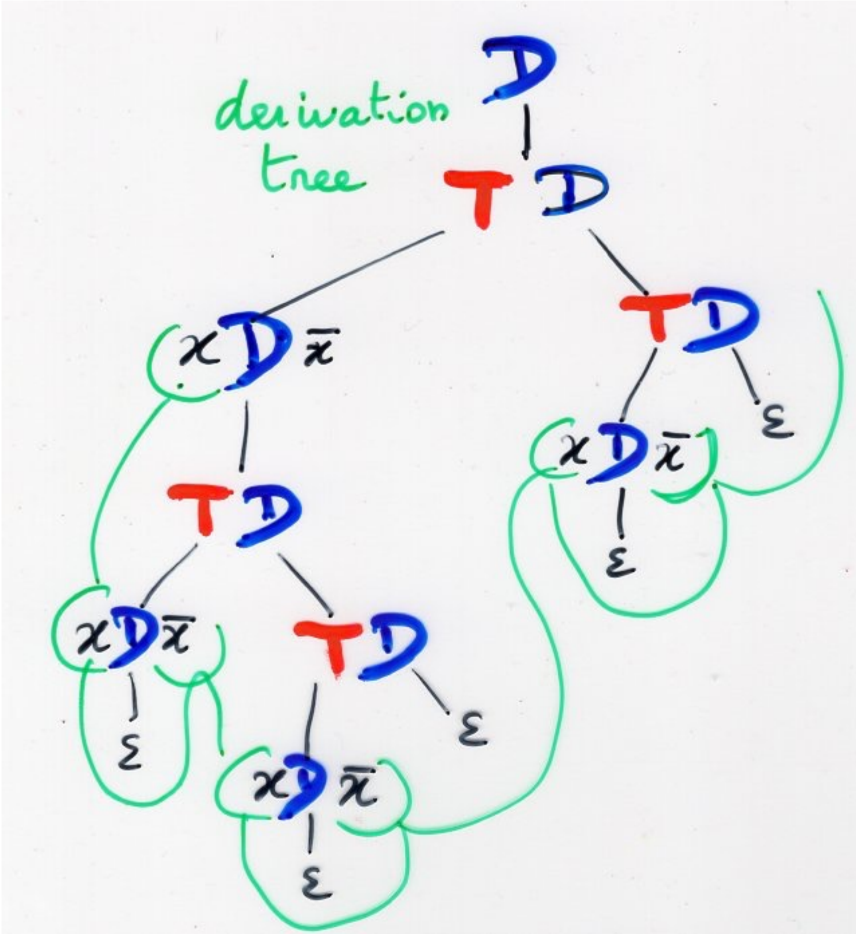


$$D$$
$$T\,D$$
$$x\,D\,\bar{x}\,D$$
$$x\,T\,D\,\bar{x}\,D$$
$$x\,x\,D\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,T\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,D\,\bar{x}\,D\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,T\,D$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,D\,\bar{x}$$
$$x\,x\,\bar{x}\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}$$

algebraic grammar

$D \to TD$ or $D \to \varepsilon$ (empty word)

$T \to x D \bar{x}$



$x\, x\, \bar{x}\, x\, \bar{x}\, \bar{x}\, x\, \bar{x}$

ambiguous grammar

if there exist a word $w \in L$ having 2 distinct derivation trees

$D \to DD$ or ε

$D \to x D \bar{x}$

ambiguous grammar

Prop Chomsky-Schützenberger

L algebraic language, $L \subseteq X^*$
having a non-ambiguous grammar.
let $a_n = |L \cap X^n|$ (nb of words of L
of length n)

Then the power series $f_L = \sum_{n \geq 0} a_n t^n$
is algebraic
(on $\mathbb{Q}(x)$)

$$\underline{\underline{D}} = \sum_{w \in D} w \quad \in \mathbb{K} \langle\langle X \rangle\rangle$$
$X = \{x, \bar{x}\}$

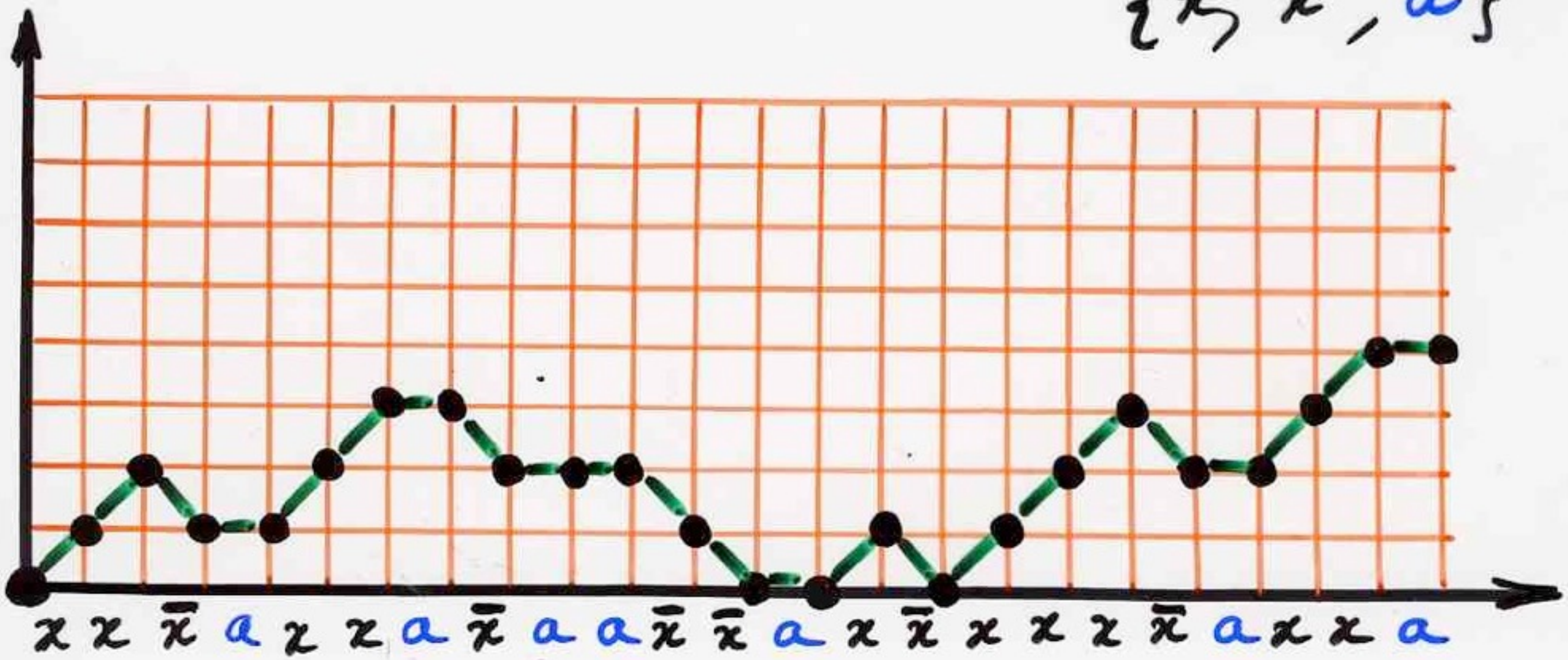
characteristic
non-commutative power series

$$\underline{\underline{D}} = 1 + x \underline{\underline{D}} \bar{x} \underline{\underline{D}}$$

<u>exercise</u> algebraic equations for Motzkin paths and prefix of Motzkin paths

prefix (left factor) of a Motzkin path (word)

$\{x, \bar{x}, a\}$



Rational langage

words accepted by a finite automaton

initial state $s_0 \in S$

$$\mathcal{A} = (S , X , \theta , s_0 , F)$$

states alphabet transition function final states $F \subset S$

$$\theta : (s, x) \longrightarrow t \in S$$

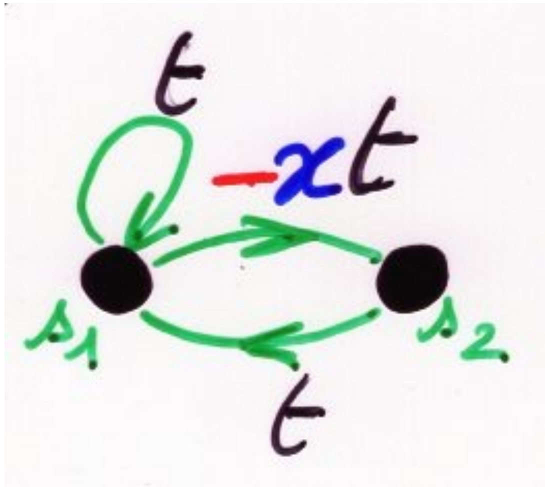
analog of transition matrix methodology

Rational $\cap$ Algebraic languages

$= n$

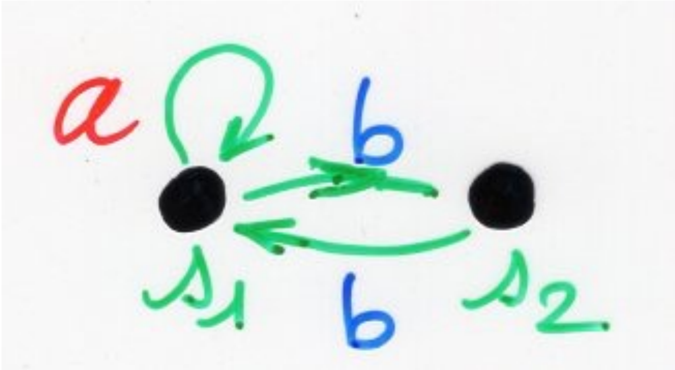
bijection

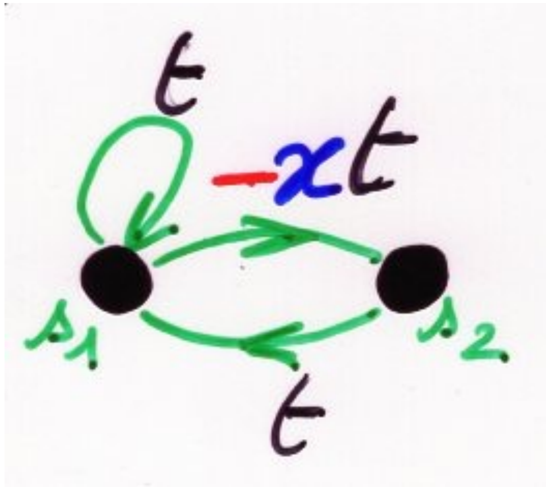
matchings of $[1, n]$ $\longleftrightarrow$ paths ω length n going from s_1 to s_1



such that
$$v(\omega) = (-x)^k t^n$$
k = number of dimers of the matching

$= n$





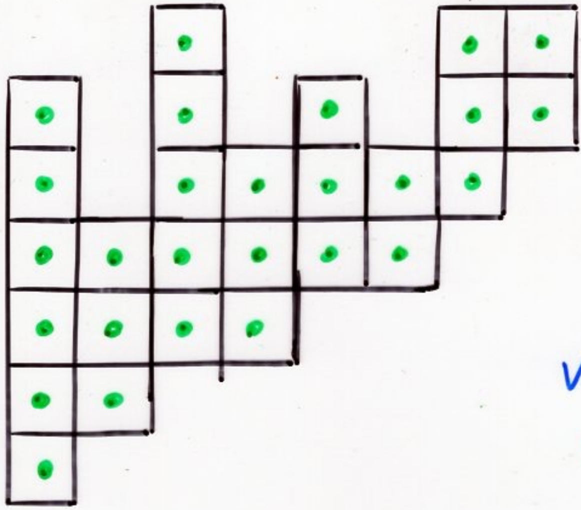
initial state s_1
final states $F = \{s_1\}$

$$X = \{a, b\}^*$$

$$L = (a + bb)^*$$

(product of words
a and bb)

partial
automaton



directed
and
vertically convex
polyomino

hint for
the exercise:

think in terms
of automaton

a_n number of such polyominoes
with n cells

$a_n = F_{2n-2}$ Fibonacci numbers

generating functions

algebraic

D-finite

or not D-finite ?

rational power series

$\cap$

algebraic power series

$\cap$

P-recursive (D-finite) power series

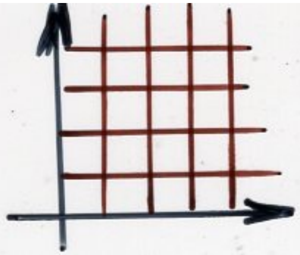
= holonomic

$$\sum_{n\geq 0} a_n t^n = \frac{N(t)}{D(t)}$$

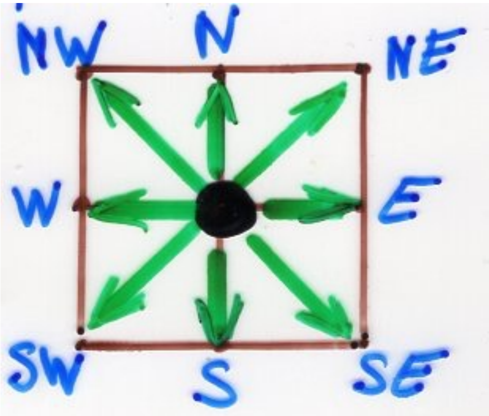
$$P(y,t) = 0$$

$$P_k(n)a_{n+k} + P_{k-1}(n)a_{n+k-1} + \dots + P_0(n)a_n = 0$$

rational power series $\Leftrightarrow$ recurrence relation with $P_0, \dots, P_k$ constants example paths in a quadrant $\mathbb{N}\times\mathbb{N}$



elementary steps

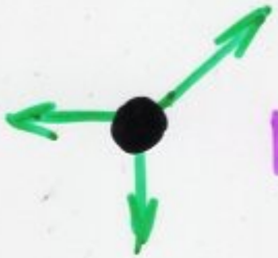


reduced to 79 cases

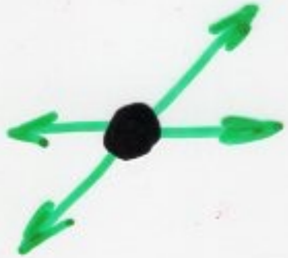
23 are D-finite

56 not D-finite

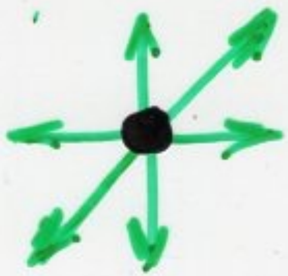
4 are algebraic



Kreweras



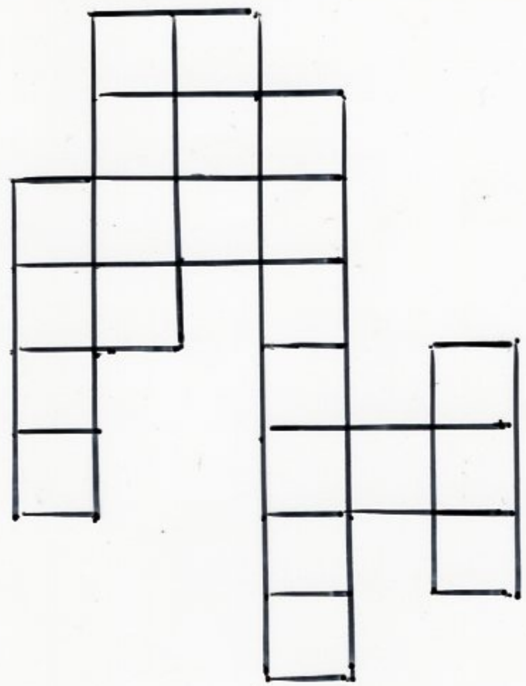
Gessel



4 are algebraic

Introduction of catalytic variables
kernel methodology

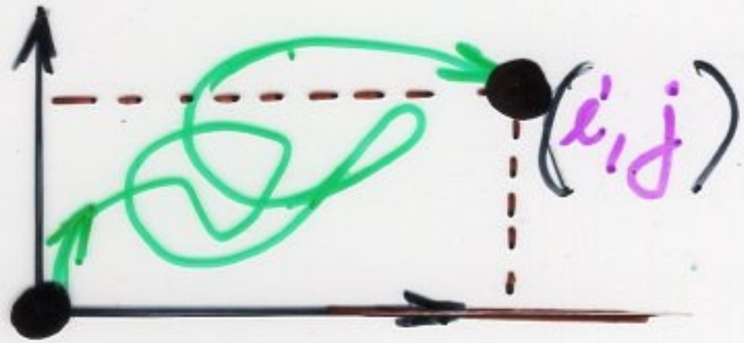
example:- vertically convex polyominoes
enumerated by perimenter, area
and number of cells in the last column
catalytic variable



- planar maps (Tutte)

one catalytic variable $\longrightarrow$ algebraic

Two catalytic variables



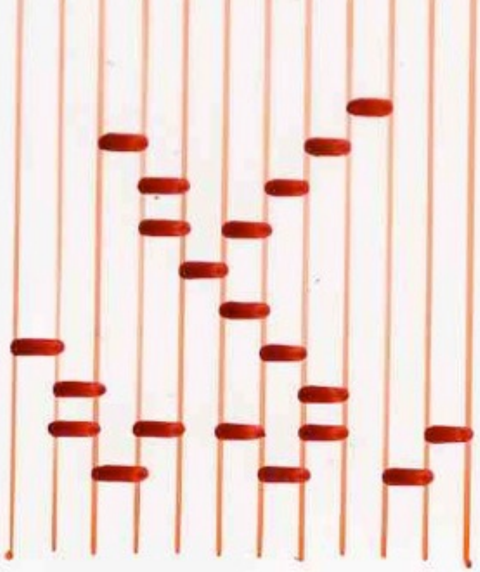
reduced to 79 cases

23 are D-finite
56 not D-finite

M. Bousquet-Mélou

Group associated to a path

D-finite $\Leftrightarrow$ the group is finite



connected heap of dimers = no empty column

$C(t)$ g.f. connected heap

(Bousquet-Mélou, Rechnitzer) 2002

$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}\right]}$$

not D-finite

$$Q = \sum_{n\geq 1} C_n t^n$$

Catalan number