Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 4

Expanding a power series into continued fraction

Chapter 4a

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Chapter 4

equivalently:

computing the coefficients

$\lambda_k \qquad b_k$

of the 3-terms linear recurrence knowing
the moments of the orthogonal polynomials

From the moments

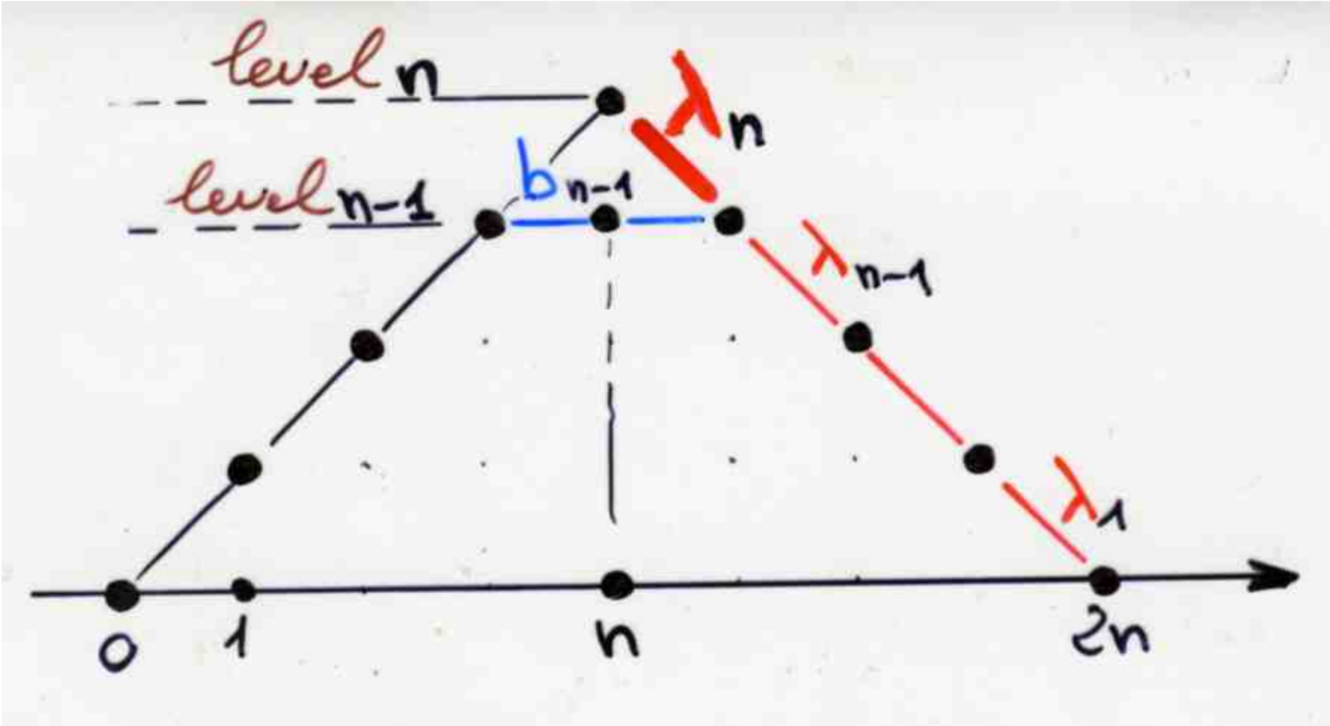
to

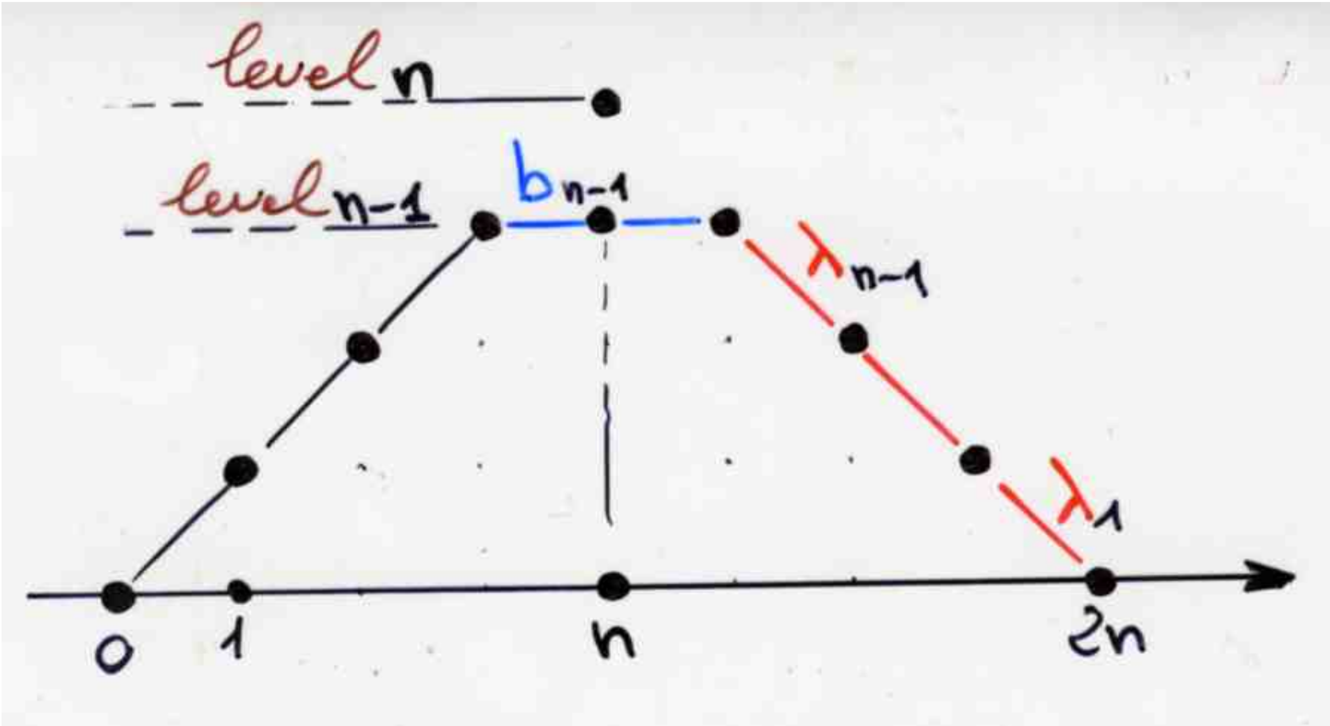
$\{\mu_n\}_{n\geqslant 0}$

moments

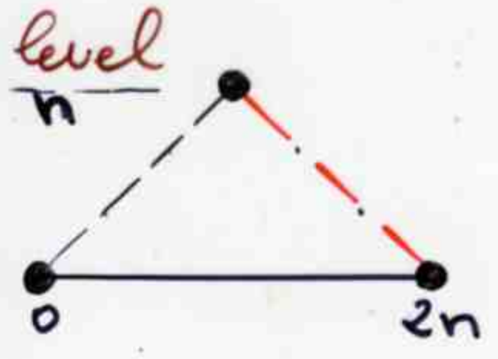
$\{b_k\}_{k\geqslant 0}, \ \{\lambda_k\}_{k\geqslant 1}$

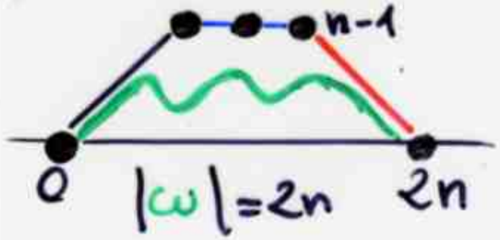
an algorithm with paths

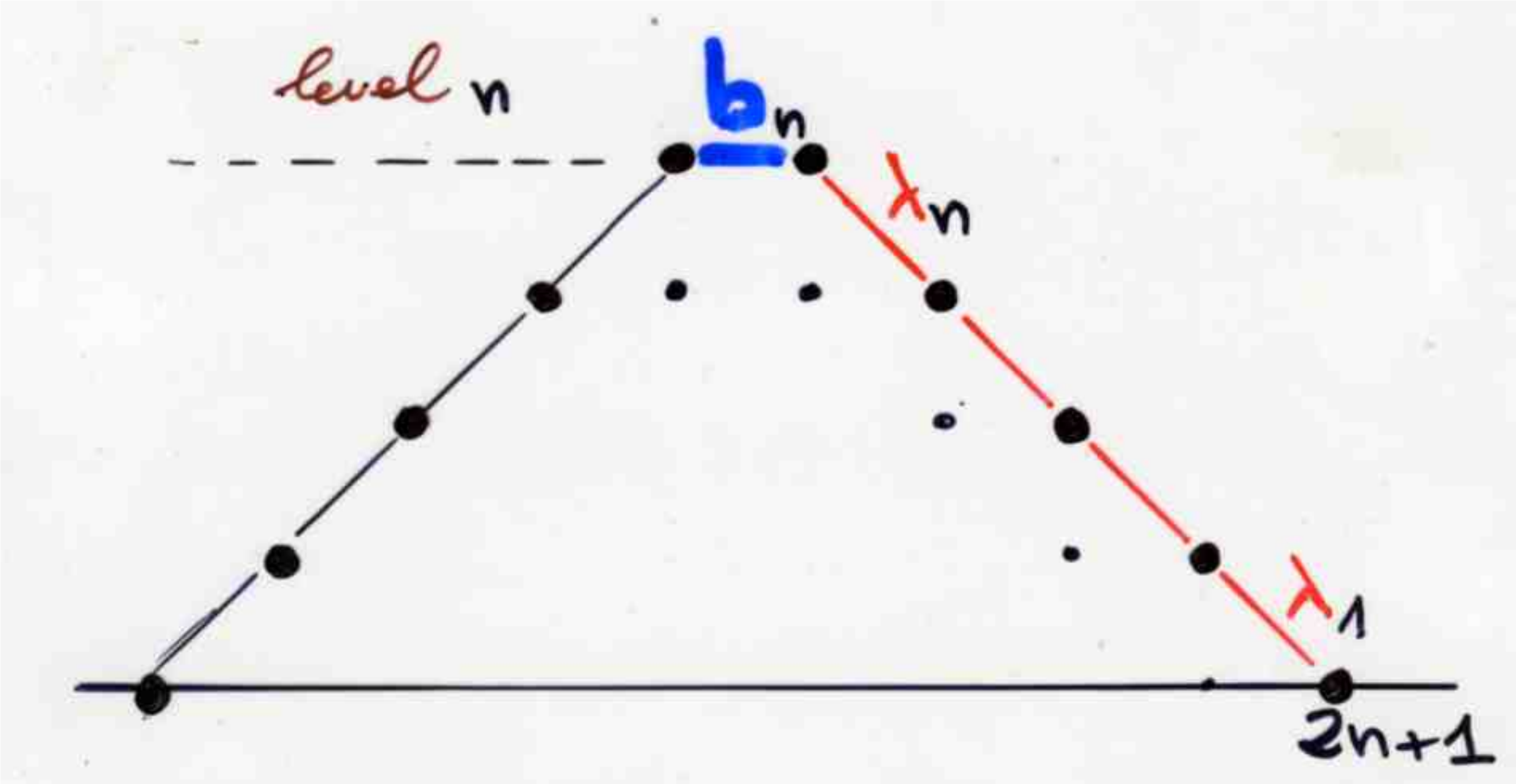

level n
level n-1
b_{n-1}
λ_n
λ_{n-1}
λ_1
0
1
n
2n



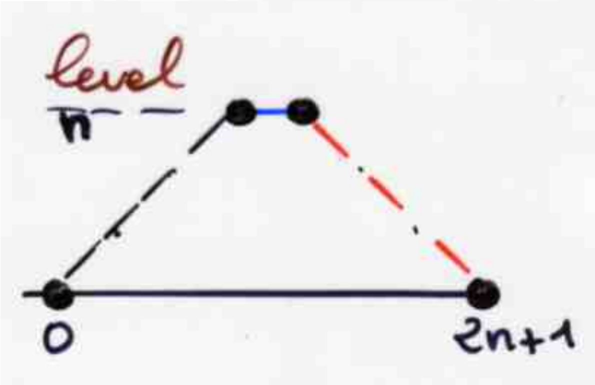
$$\mu_{2n} = \lambda_1 \cdots \lambda_n + \sum_{\omega \,.\, \text{Motzkin path}} v(\omega)$$

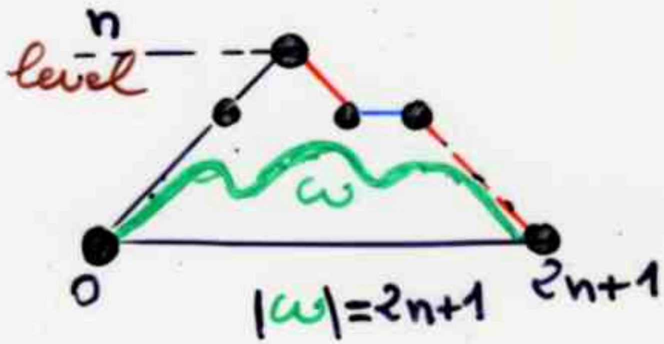





level n
b_n
λ_n
λ_1
$2n+1$

$$\mu_{2n+1} = \lambda_1 \; \lambda_n \, b_n + \sum_{\omega \text{ Motzkin path}} v(\omega)$$



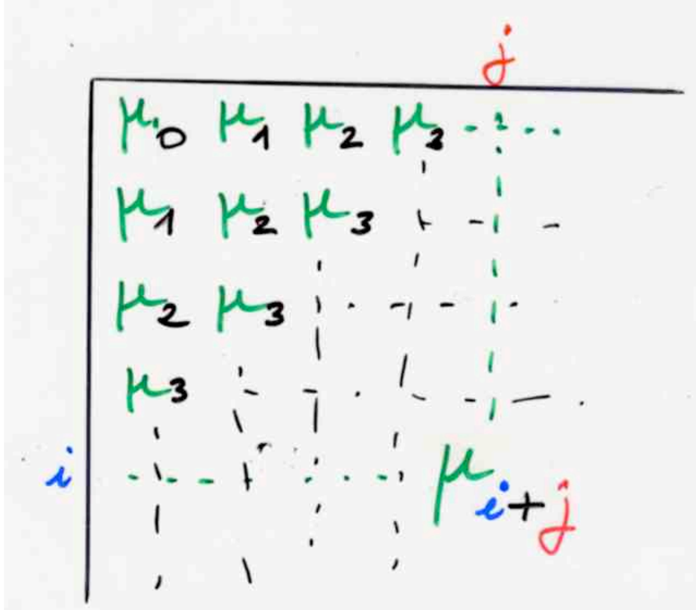


Hankel determinants

Hankel determinant

any minor of the matrix

$\mathbf{H}\left(\{\mu_n\}_{n\geq 0}\right)$



LGV Lemma

determinant $\longleftrightarrow$ configuration of non-intersecting paths

Toeplitz matrix

c_{i-j}

The LGV Lemma

Part I, Ch5a, 3-28

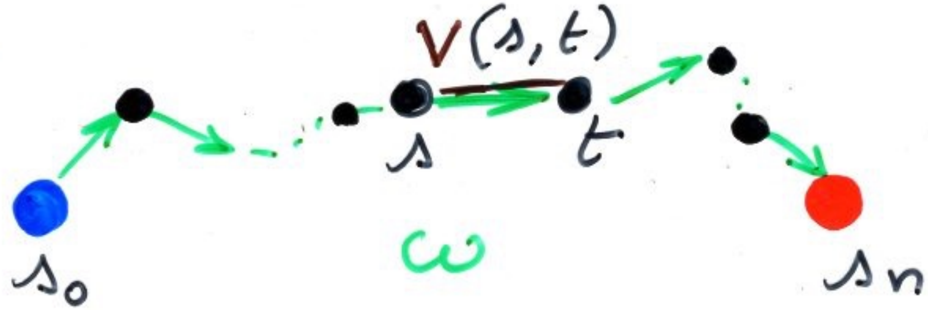

non-intersecting configuration of paths
determinant

Path $\omega = (s_0, s_1, \dots, s_n)$ $s_i \in S$

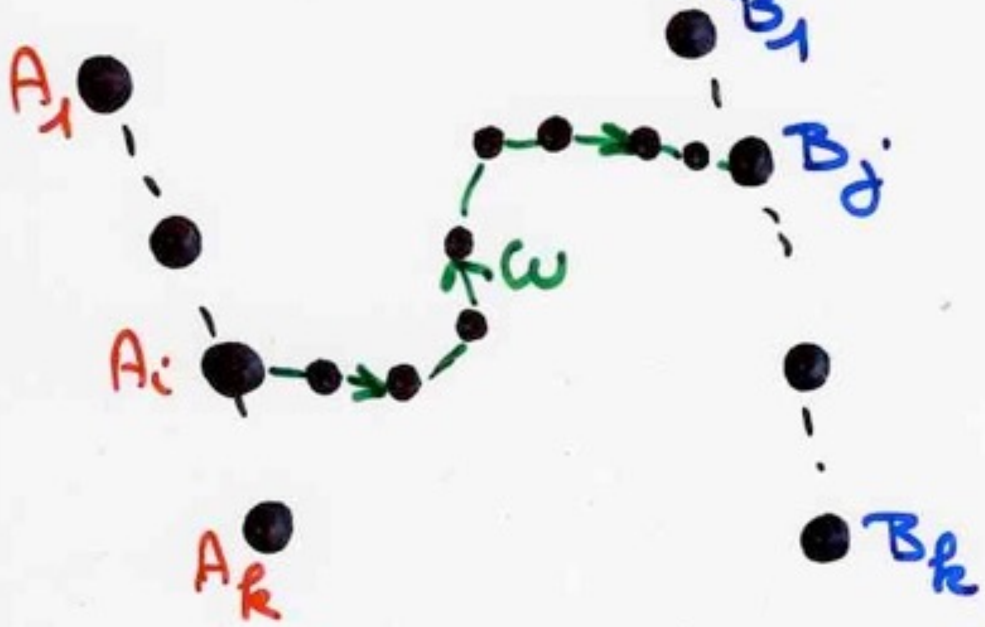
<u>notation</u> $s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation $v : S \times S \longrightarrow \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \cdots v(s_{n-1}, s_n)$$



weighted path



$A_1, \dots, A_k$
$B_1, \dots, B_k$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \dots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \ldots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \ldots (\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting

Proof: Involution ϕ

$$E = \left\{ (\sigma; (\omega_1, \dots, \omega_k)) \; ; \; \begin{array}{l} \sigma \in S_n \\ \omega_i : A_i \rightsquigarrow B_{\sigma(i)} \end{array} \right\}$$

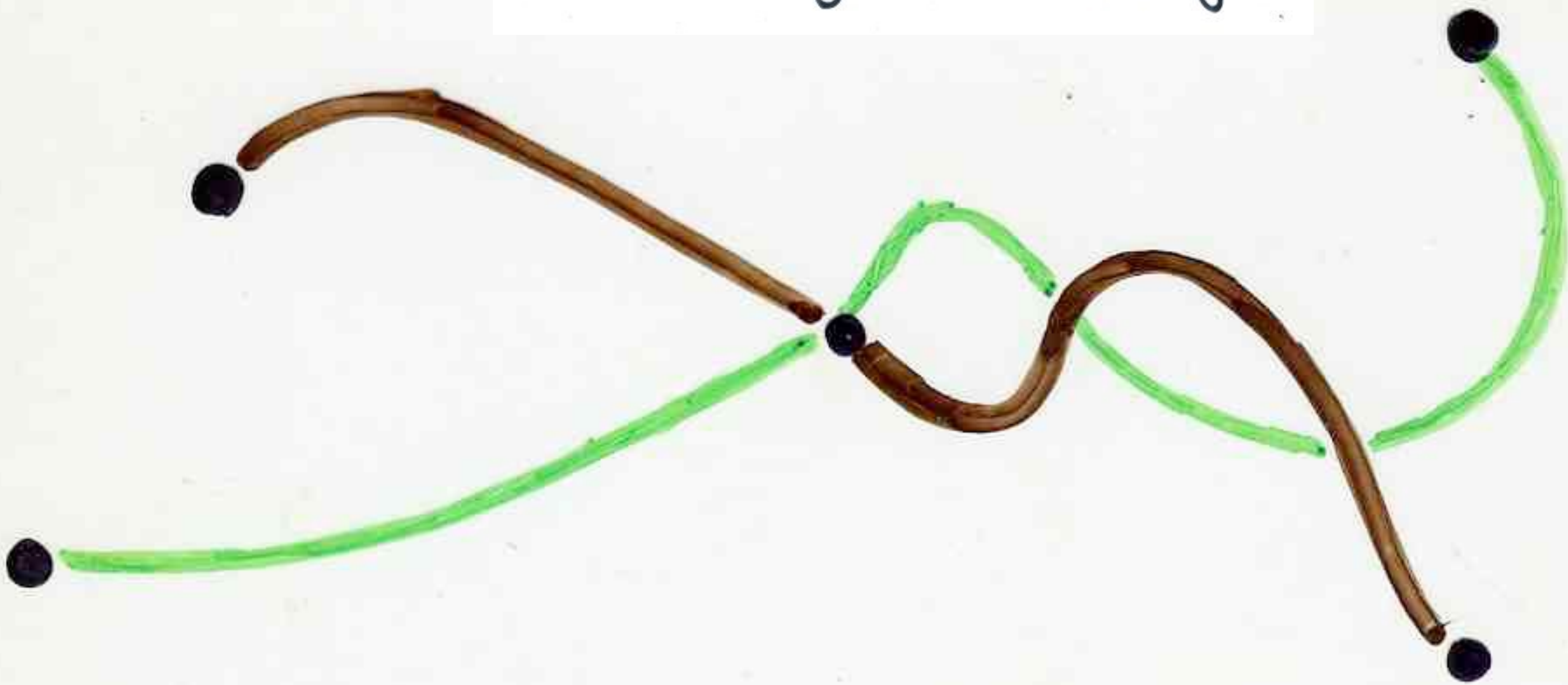
$NC \subseteq E$ non-crossing configurations

$$\phi : (E - NC) \to (E - NC)$$

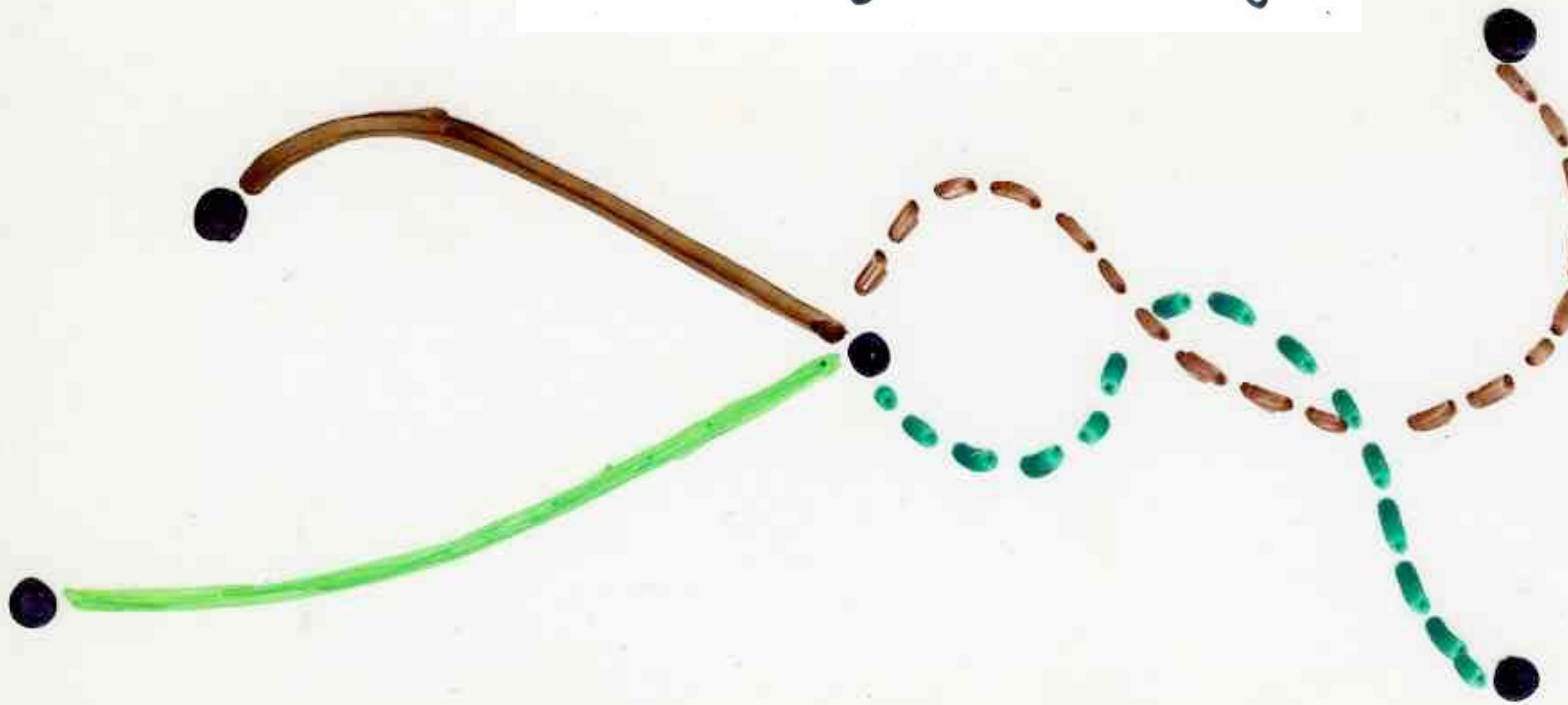
$$\phi(\sigma; (\omega_1, \dots, \omega_k)) = (\sigma'; (\omega'_1, \dots, \omega'_k))$$

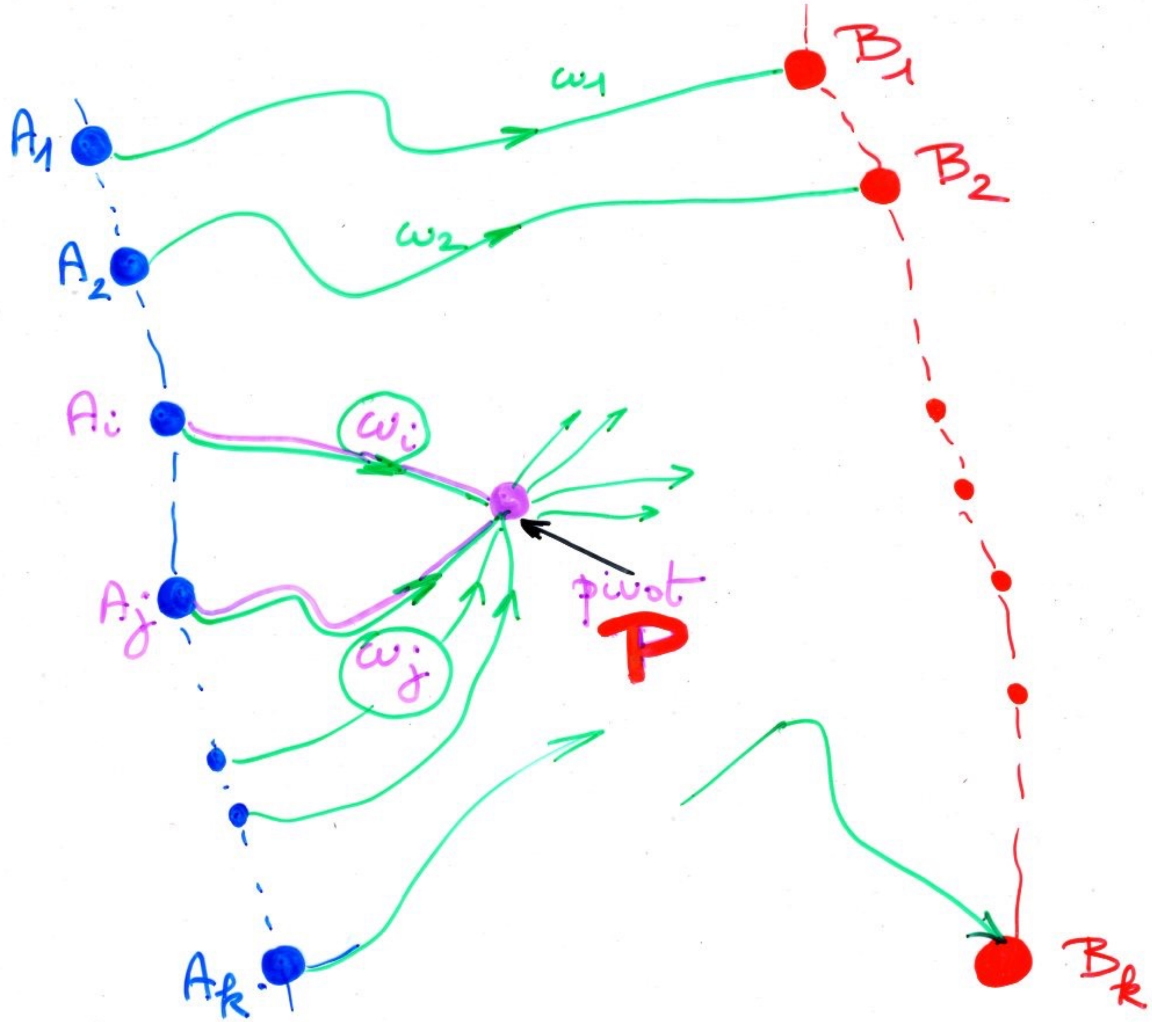
$$\begin{cases} (-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')} \\ v(\omega_1) \dots v(\omega_k) = v(\omega'_1) \dots v(\omega'_k) \end{cases}$$

idea of the proof:



idea of the proof:

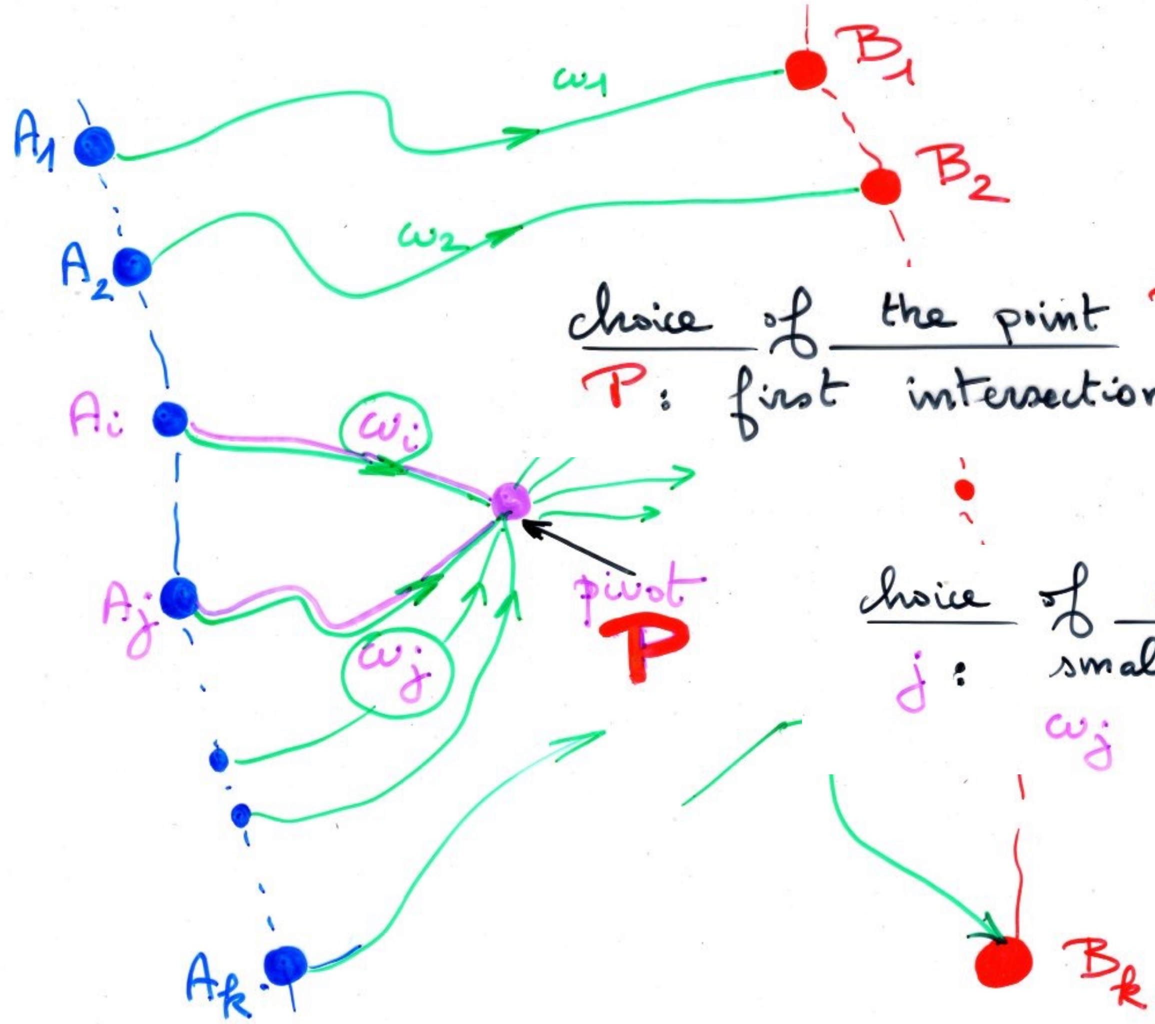




A_1
A_2
A_i
A_j
A_k
B_1
B_2
B_k
ω_1
ω_2
ω_i
ω_j
pivot
P

choice of ω_i

i : smallest i, $1 \leq i \leq k$, such that ω_i has an intersection with another path



choice of the point P

P : first intersection point on the path ω_i

choice of ω_j

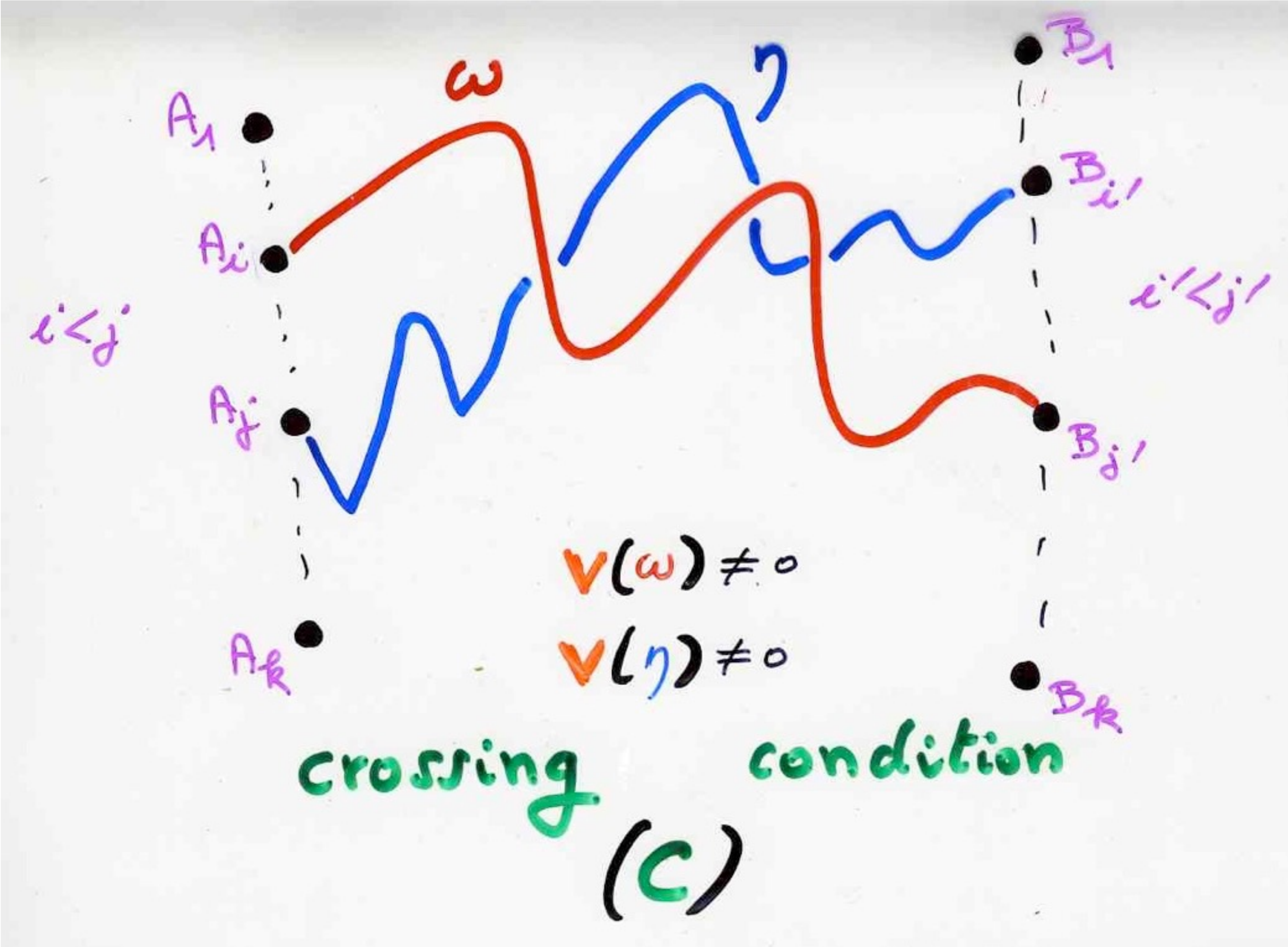
j : smallest j, $i < j \leq k$ such that ω_j intersect ω_i

LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \dots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$

paths non-intersecting



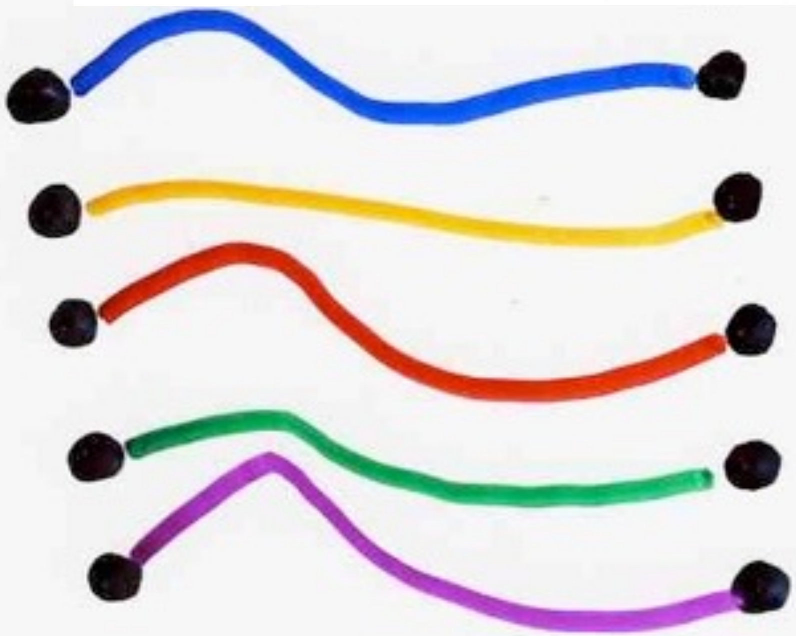
A_1
A_i
A_j
A_k
$i<j$
ω
η
B_1
$B_{i'}$
$B_{j'}$
B_k
$i'<j'$
$v(\omega) \neq 0$
$v(\eta) \neq 0$
crossing condition
(C)

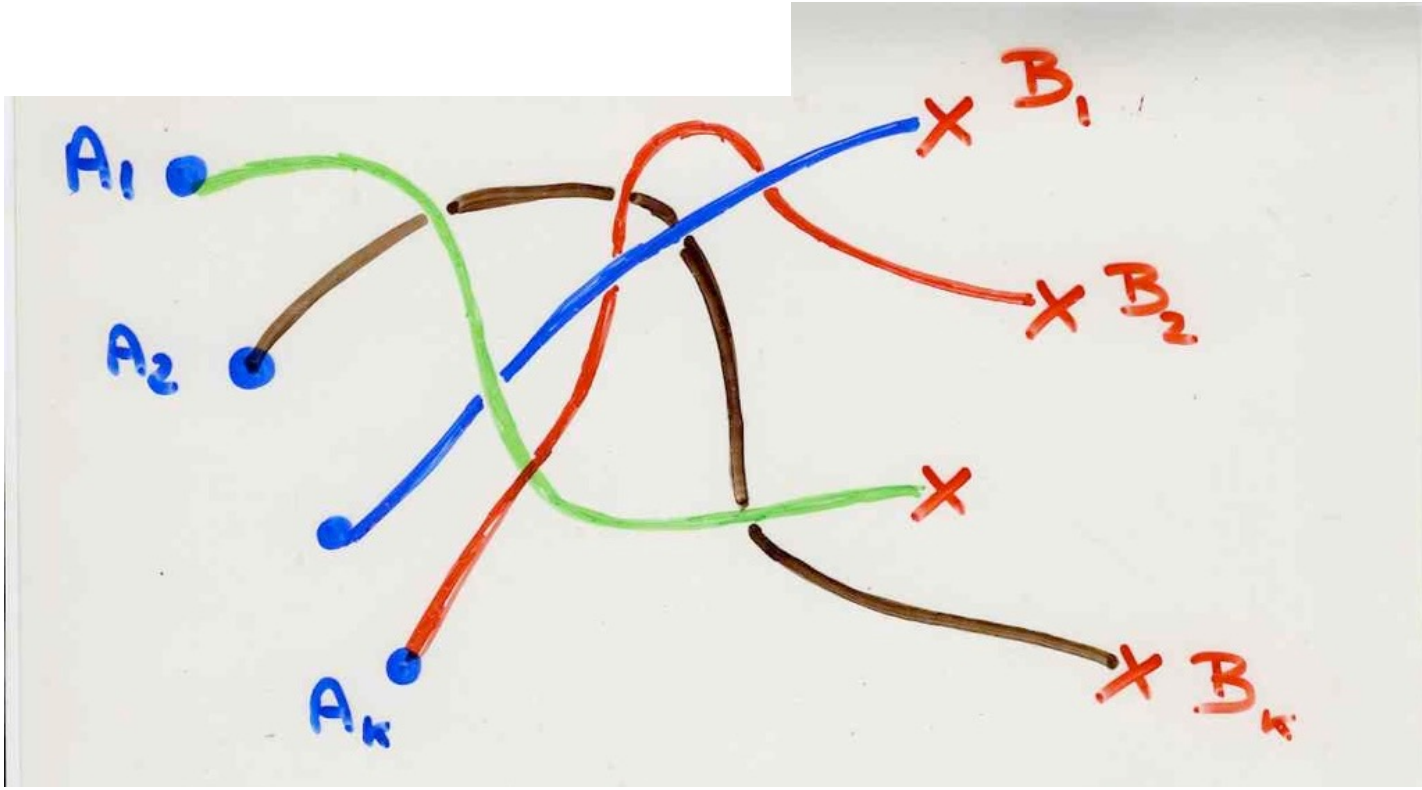
Proposition (LGV Lemma) (C) crossing condition

$$\det(a_{ij}) = \sum_{(\omega_1,\dots,\omega_k)} v(\omega_1)\dots(\omega_k)$$

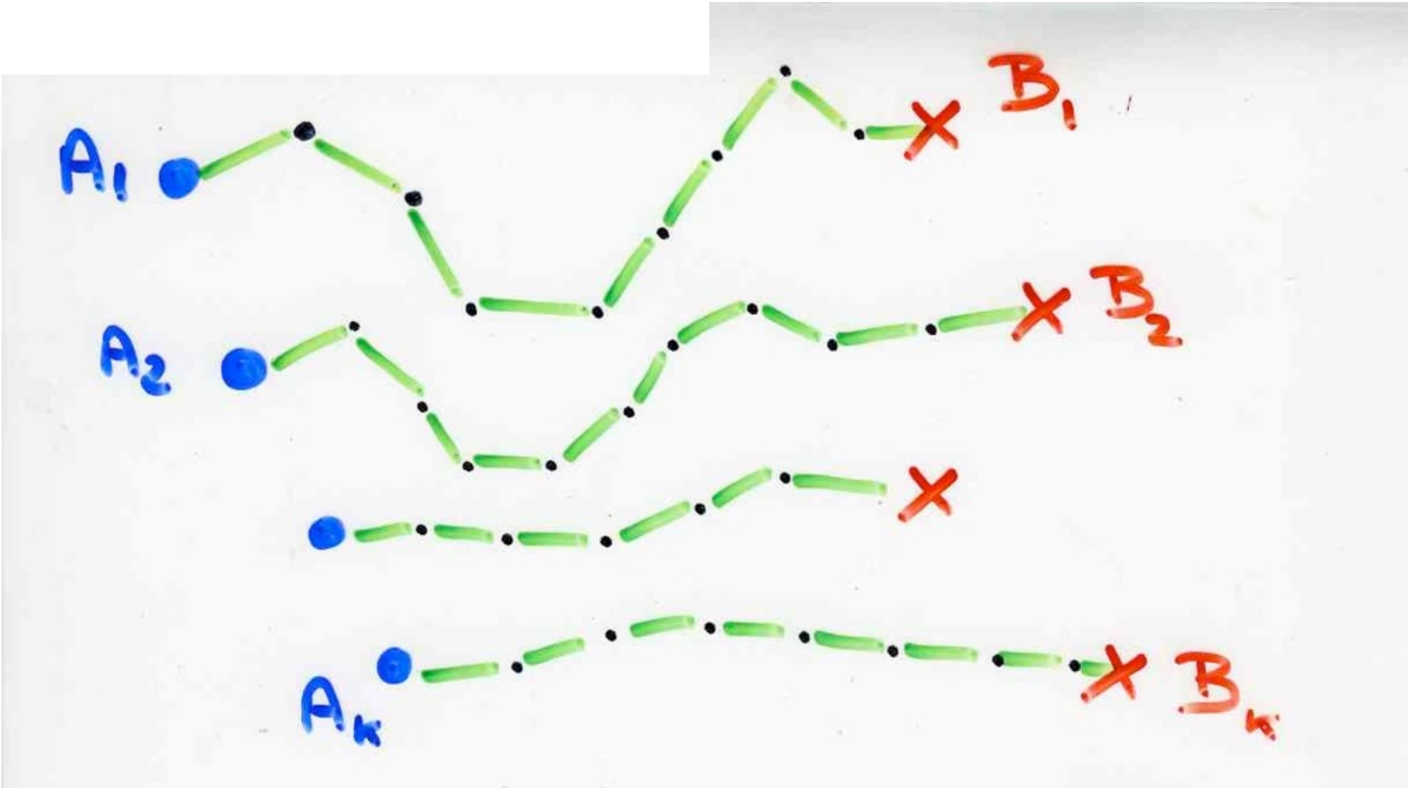
$\omega_i : A_i \rightsquigarrow B_i$

non-intersecting



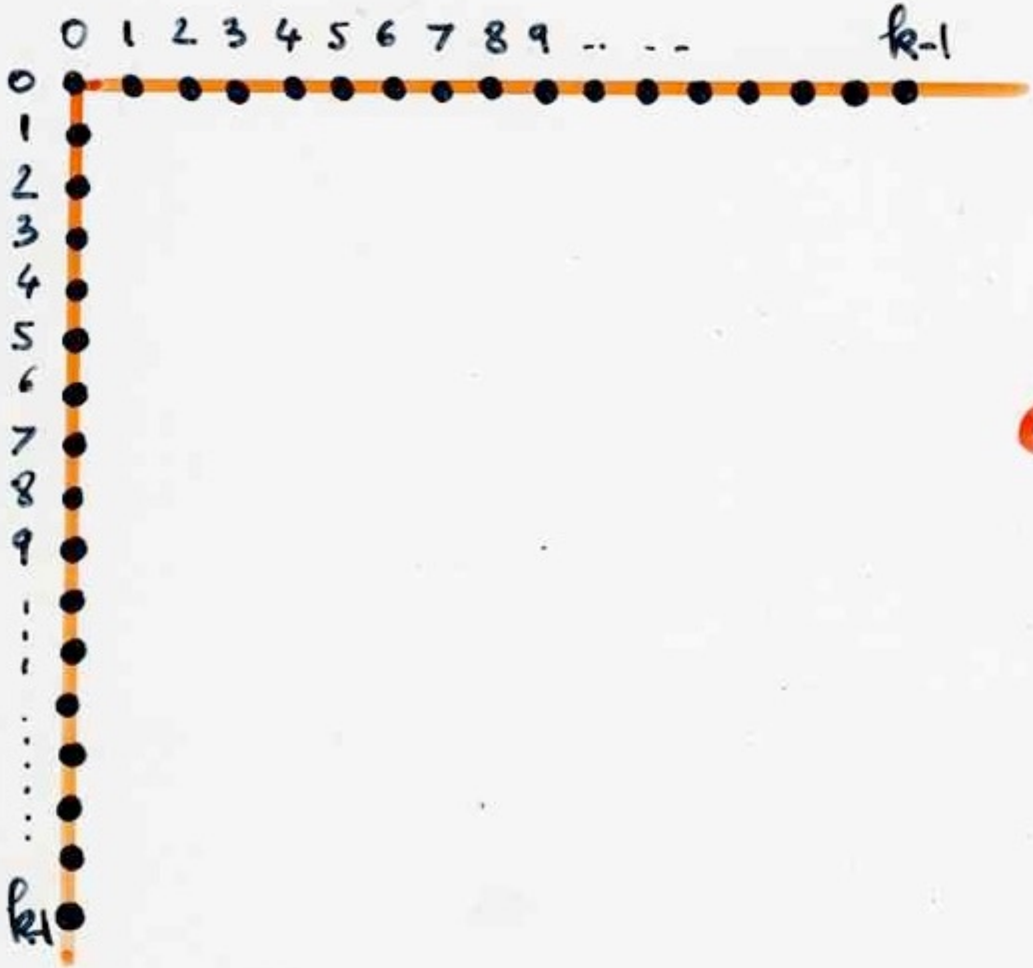


A_1
A_2
A_K
B_1
B_2
B_K

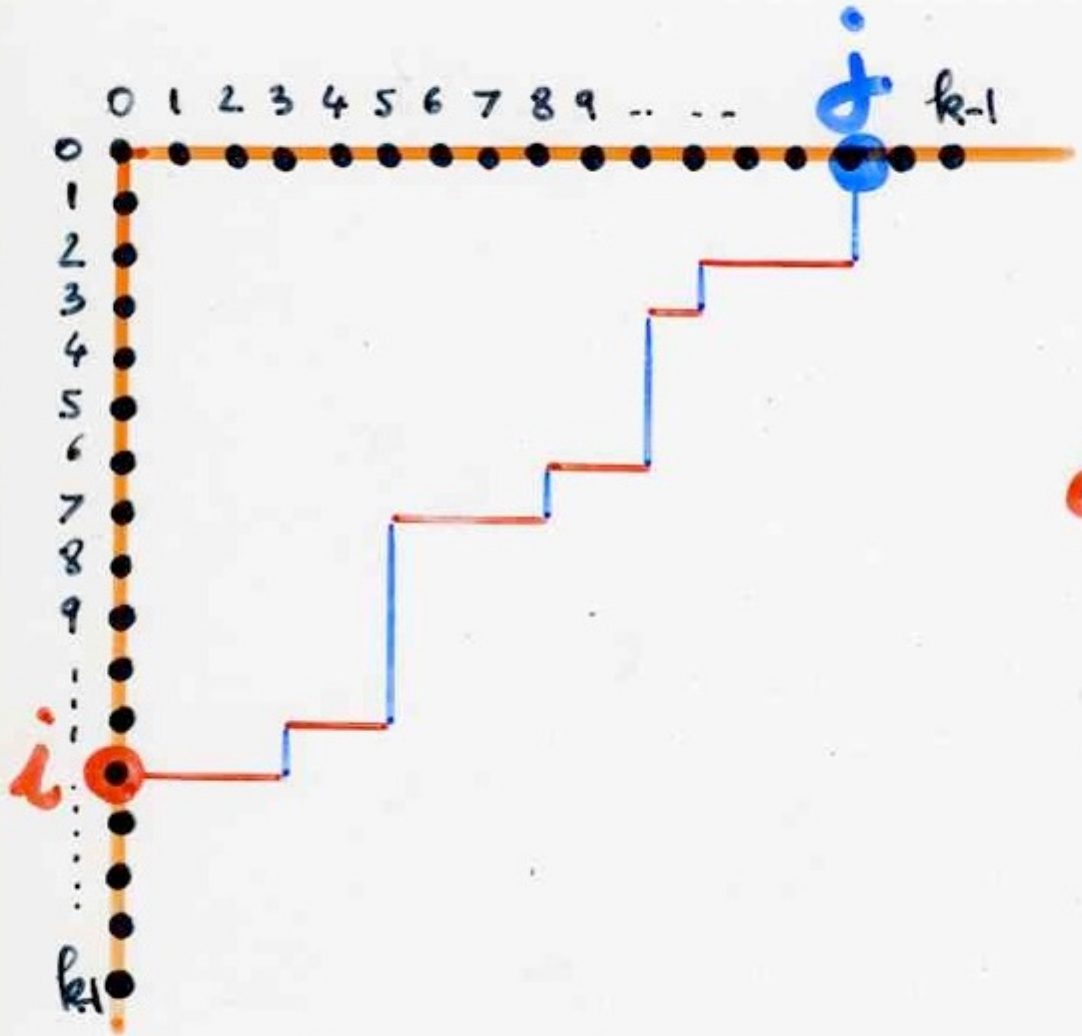


A_1
B_1
A_2
B_2
A_k
B_k

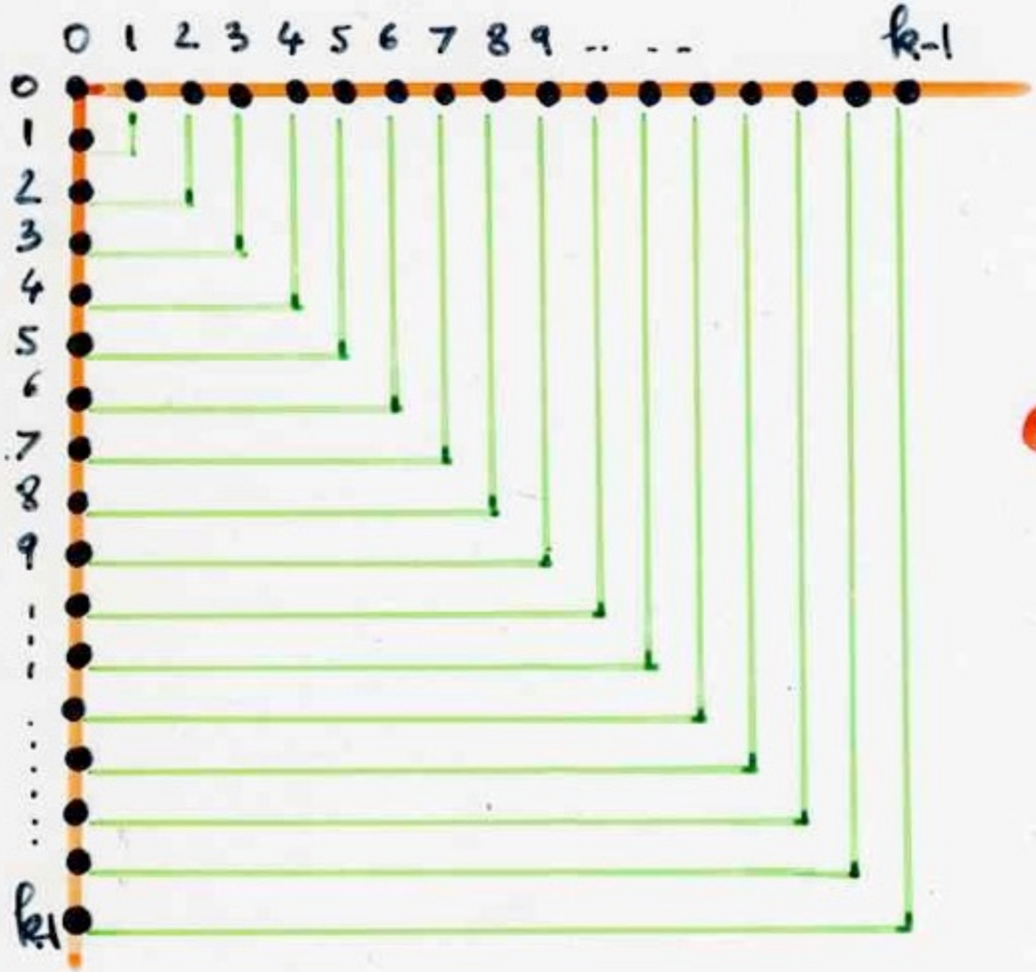
a simple example



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & \binom{i+j}{i} & \\ 1 & 5 & & & & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

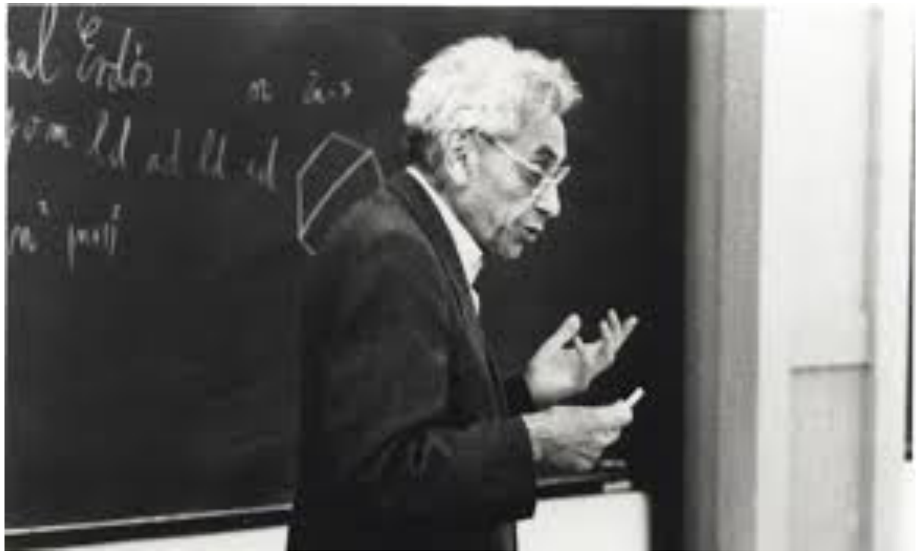
why LGV Lemma ?

Part I, Ch5a, 24-28

Lattice paths and determinants

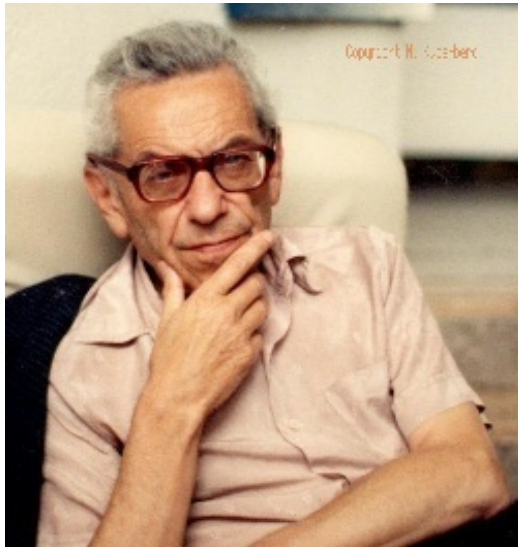
Chapter 29

Why « LGV **Lemma** » ?



Paul Erdös liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems,

Erdös also said that you need not believe in God but, as a mathematician, you should believe in The Book.



Lattice paths and determinants

Chapter 29

Why « LGV **Lemma** » ?

The essence of mathematics is proving theorems — and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside–Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma — including its proof — should be beautiful!

In this chapter we look at one such marvelous piece of mathematical reasoning, a counting lemma that first appeared in a paper by Bernt Lindström in 1972. Largely overlooked at the time, the result became an instant classic in 1985, when Ira Gessel and Gerard Viennot rediscovered it and demonstrated in a wonderful paper how the lemma could be successfully applied to a diversity of difficult combinatorial enumeration problems.

Why « **LGV** Lemma » ?

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

[4]Lindström used the term "pairwise node disjoint paths". The term "non-intersecting," which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

[5]By a curious coincidence, Lindström's result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the "Lindstrom–Gessel–Viennot theorem." It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called "Slater determinant" in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an "ancestor" of the Lindstro¨m–Gessel–Viennot determinant.

[6]There exist however also several interesting applications of the general form of the Lindstro¨m–Gessel–Viennot theorem in the literature, see [10, 16, 51].

combinatorics

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

I. Gessel and X.G.V., *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

I. Gessel and X.G.V., *Determinants, paths and plane partitions, preprint (1989)*

statistical physics: (wetting, melting)

Fisher, Vicious walkers, Botzmann lecture (1984)

combinatorial chemistry:

John, Sachs (1985)

Gronau, Just, Schade, Scheffler, Wojciechowski (1988)

probabilities, birth and death process,

Karlin , McGregor (1959)

quantum mechanics: Slater determinant

Slater(1929) (1968), De Gennes (1968)

orthogonal polynomials

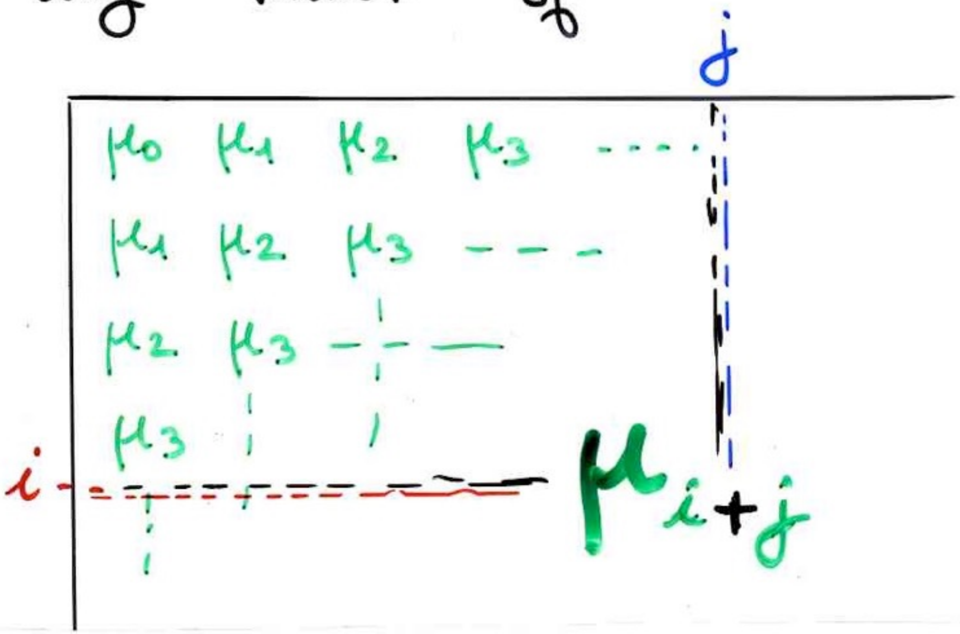
computing the coefficients

λ_k b_k

with Hankel determinants of moments

Hankel determinant

any minor of



$$H\begin{pmatrix}\alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k\end{pmatrix}$$

$$0 \leq \alpha_1 < \cdots < \alpha_k$$

$$0 \leq \beta_1 < \cdots < \beta_k$$

$$H\begin{pmatrix}\alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k\end{pmatrix}$$

$$0 \leq \alpha_1 < \cdots < \alpha_k$$
$$0 \leq \beta_1 < \cdots < \beta_k$$

$$A_i = (-\alpha_i, 0)$$

$$B_i = (\beta_i, 0)$$

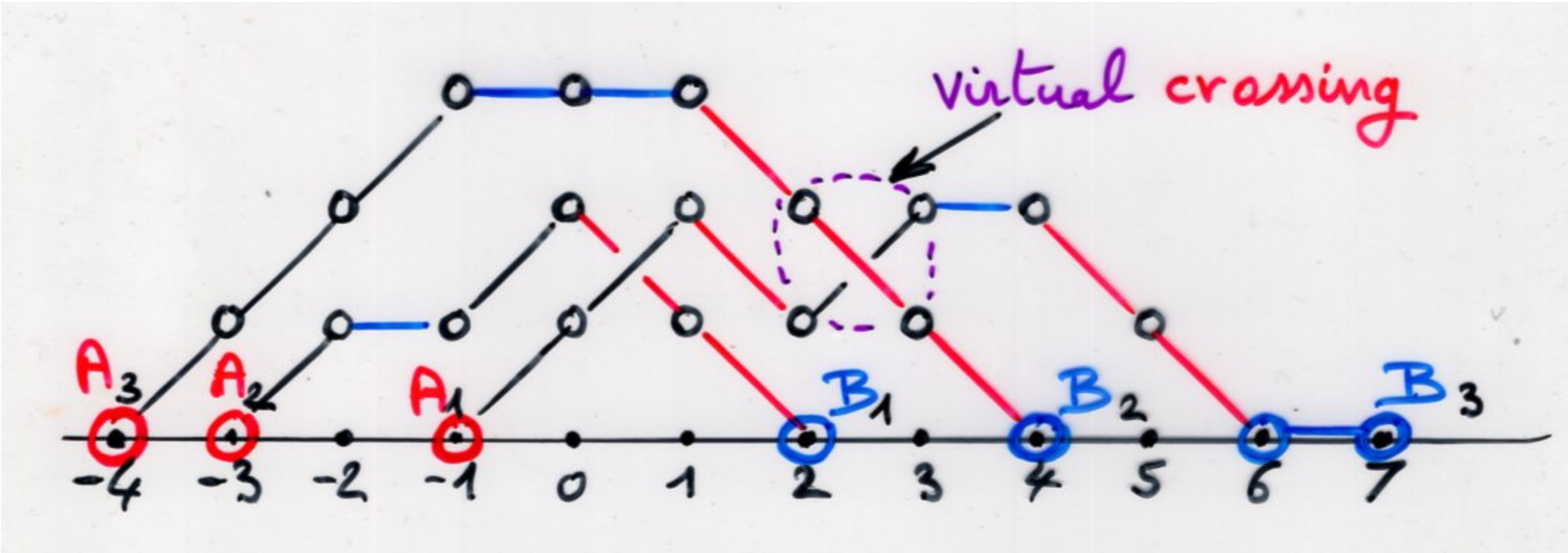
$$(1 \leq i \leq k)$$

Lemma

$$H\begin{pmatrix}\alpha_1, \dots, \alpha_k \\ \beta_1, \dots, \beta_k\end{pmatrix} = \sum_{\zeta} (-1)^{inv(\sigma)} v(\omega_1) \quad v(\omega_k)$$

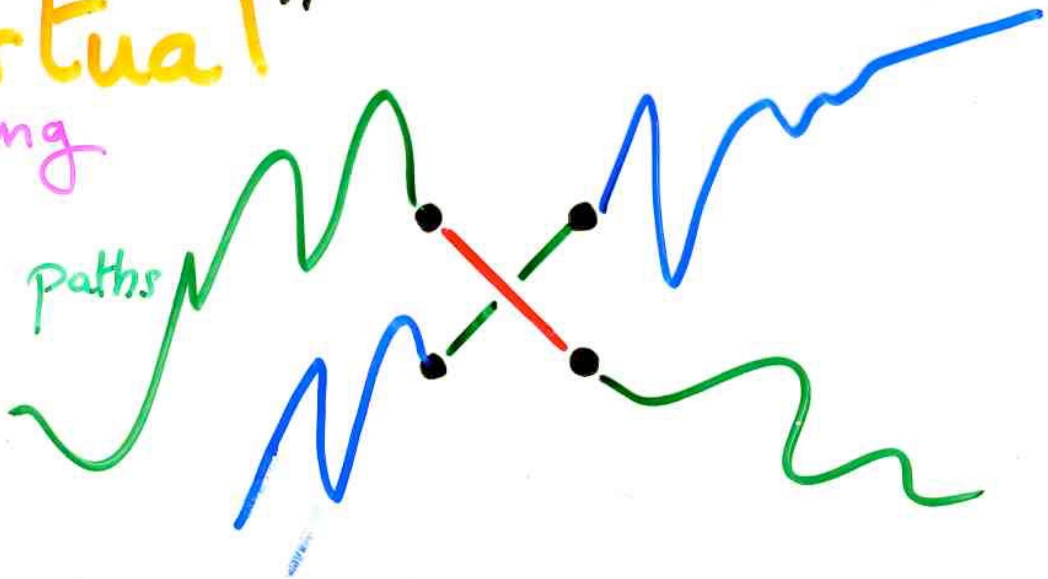
$$\zeta = (\sigma; \omega_1, \dots, \omega_k) \qquad \sigma \in \mathfrak{S}_k$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)} \qquad \{\omega_i\}_{1 \leq i \leq k} \text{ 2 by 2 disjoints}$$



$$H\begin{pmatrix} 1, & 3, & 4 \\ 2, & 4, & 7 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1, & 2, & 3 \\ 3, & 1, & 2 \end{pmatrix}$$

"virtual"
crossing
of
Motzkin paths

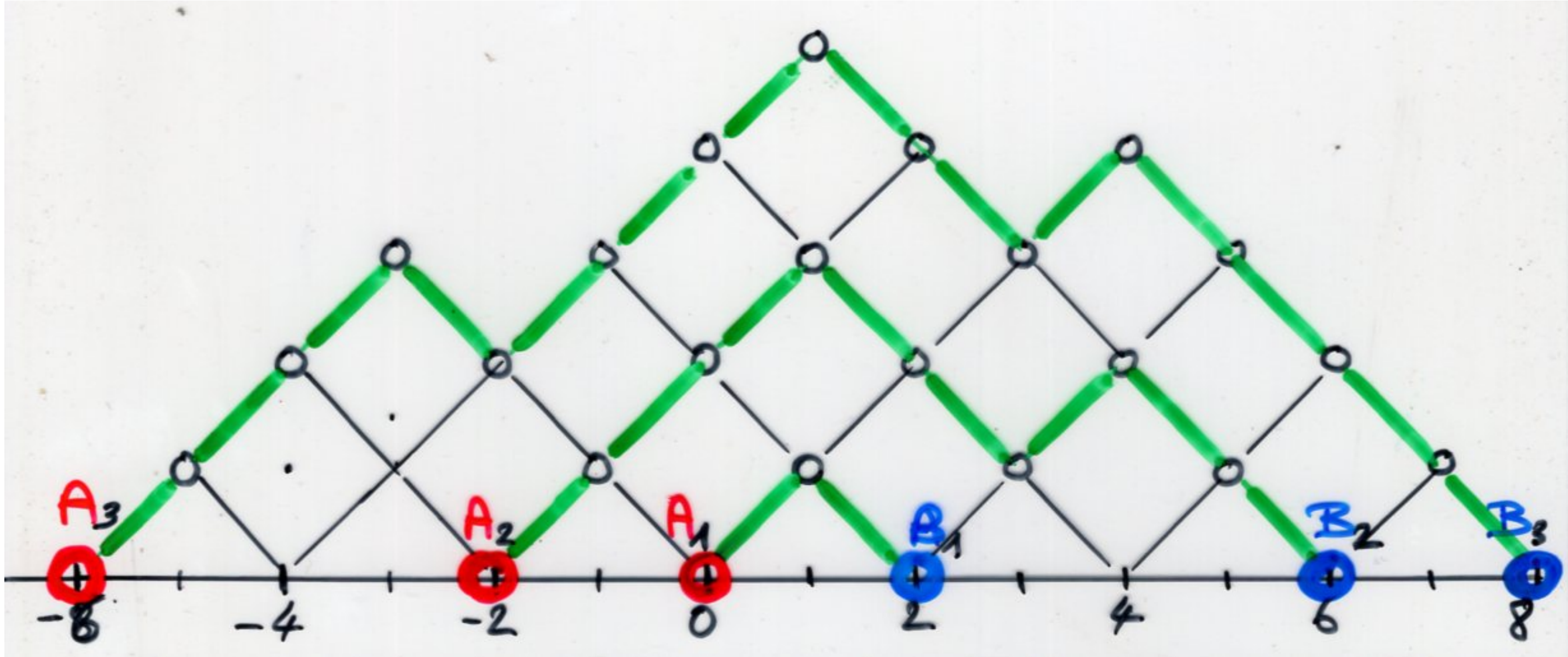


LGV Lemma. general form

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \dots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$

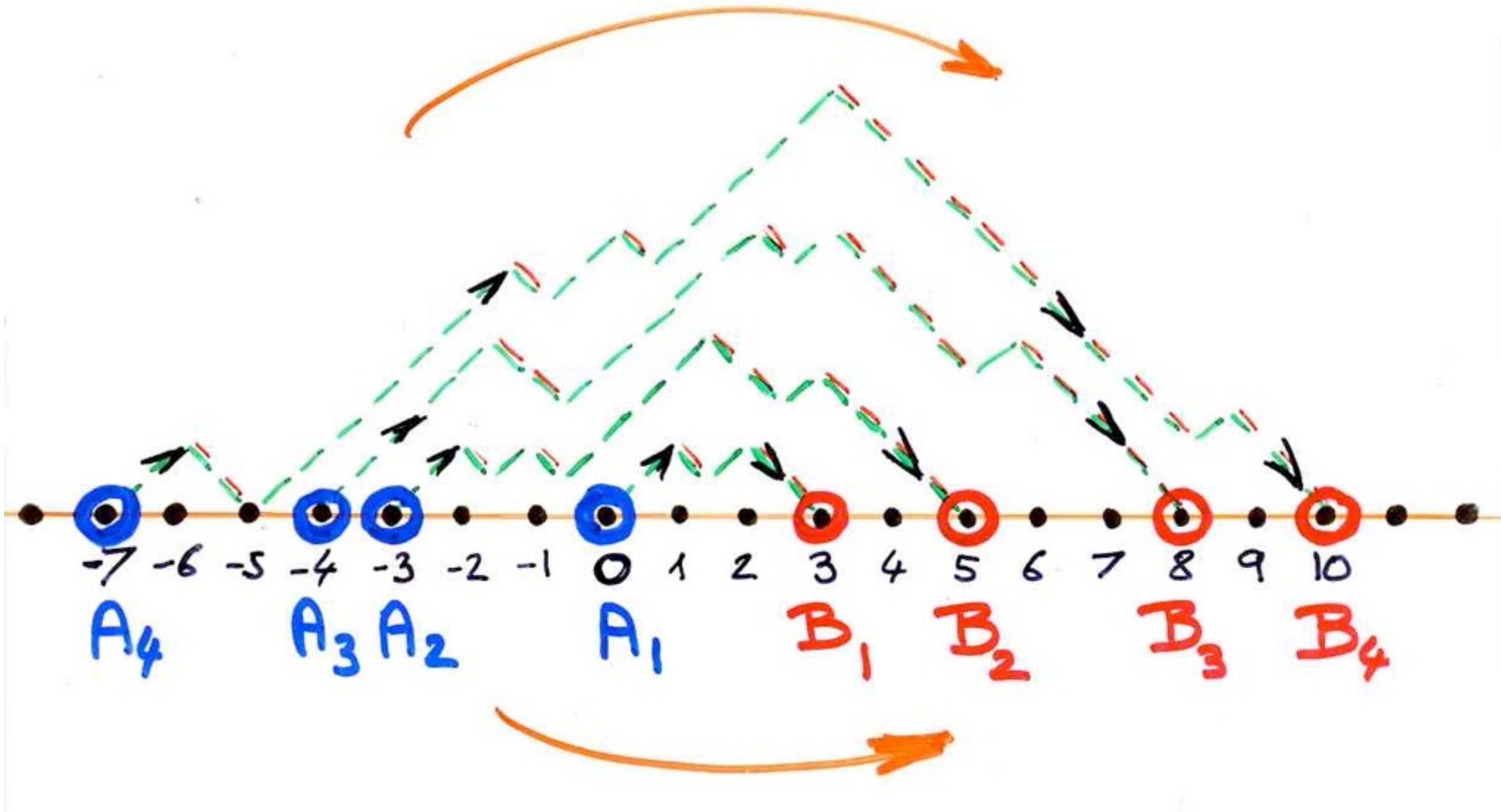
paths non-intersecting



$$H\begin{pmatrix} 0, & 2, & 6 \\ 2, & 6, & 8 \end{pmatrix}$$

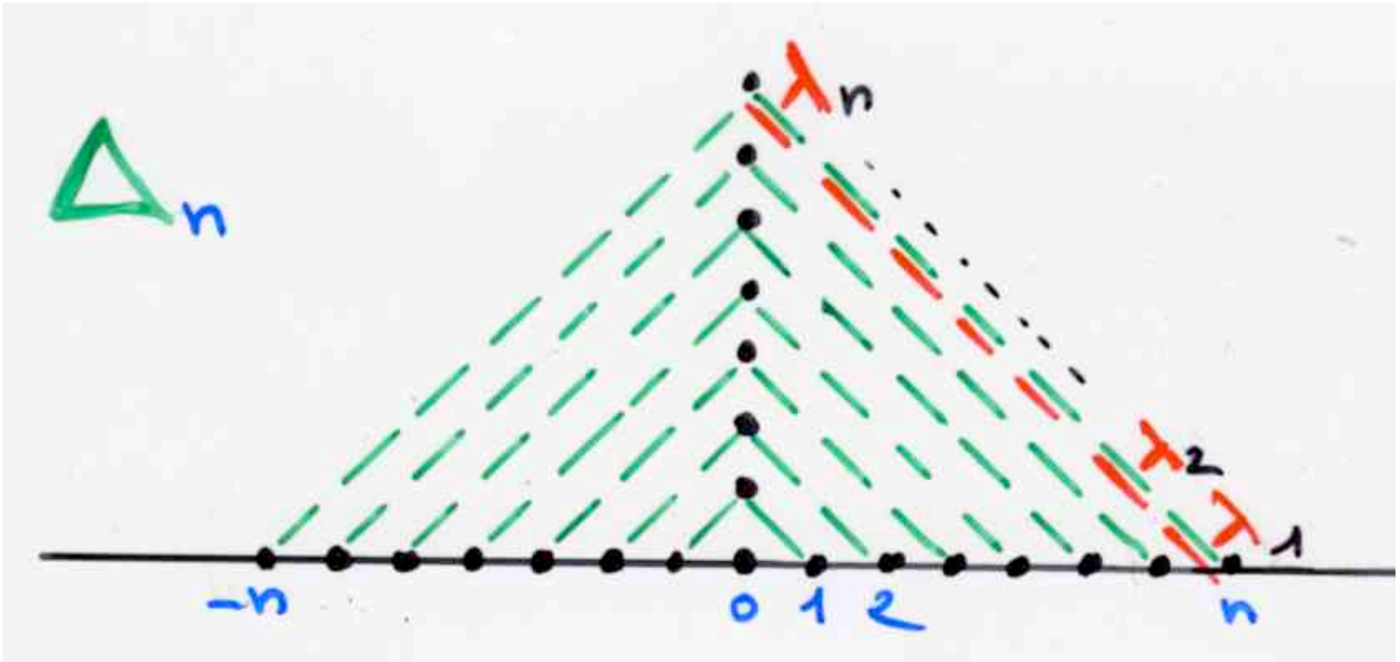
$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$

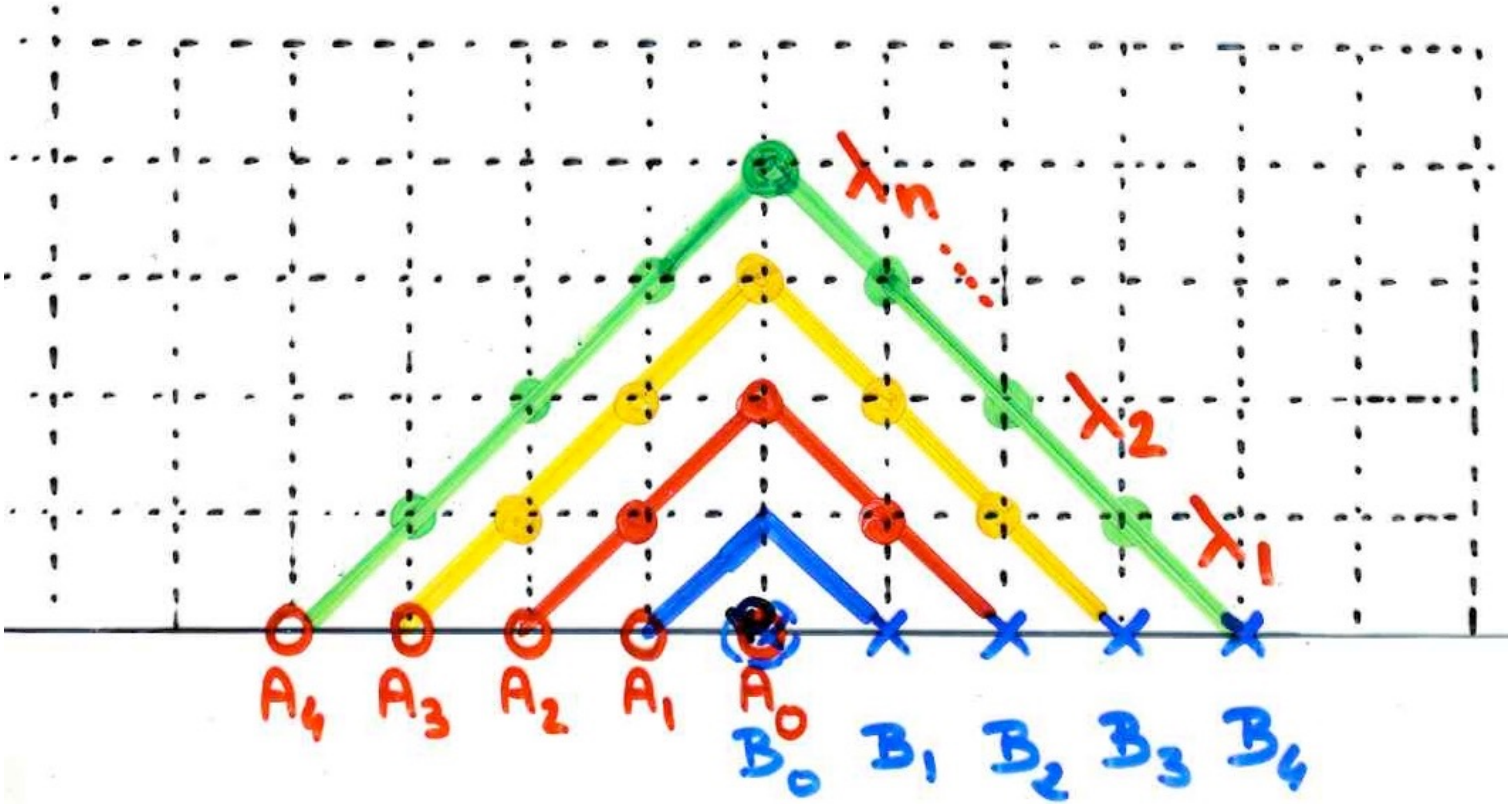
Dyck paths



$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

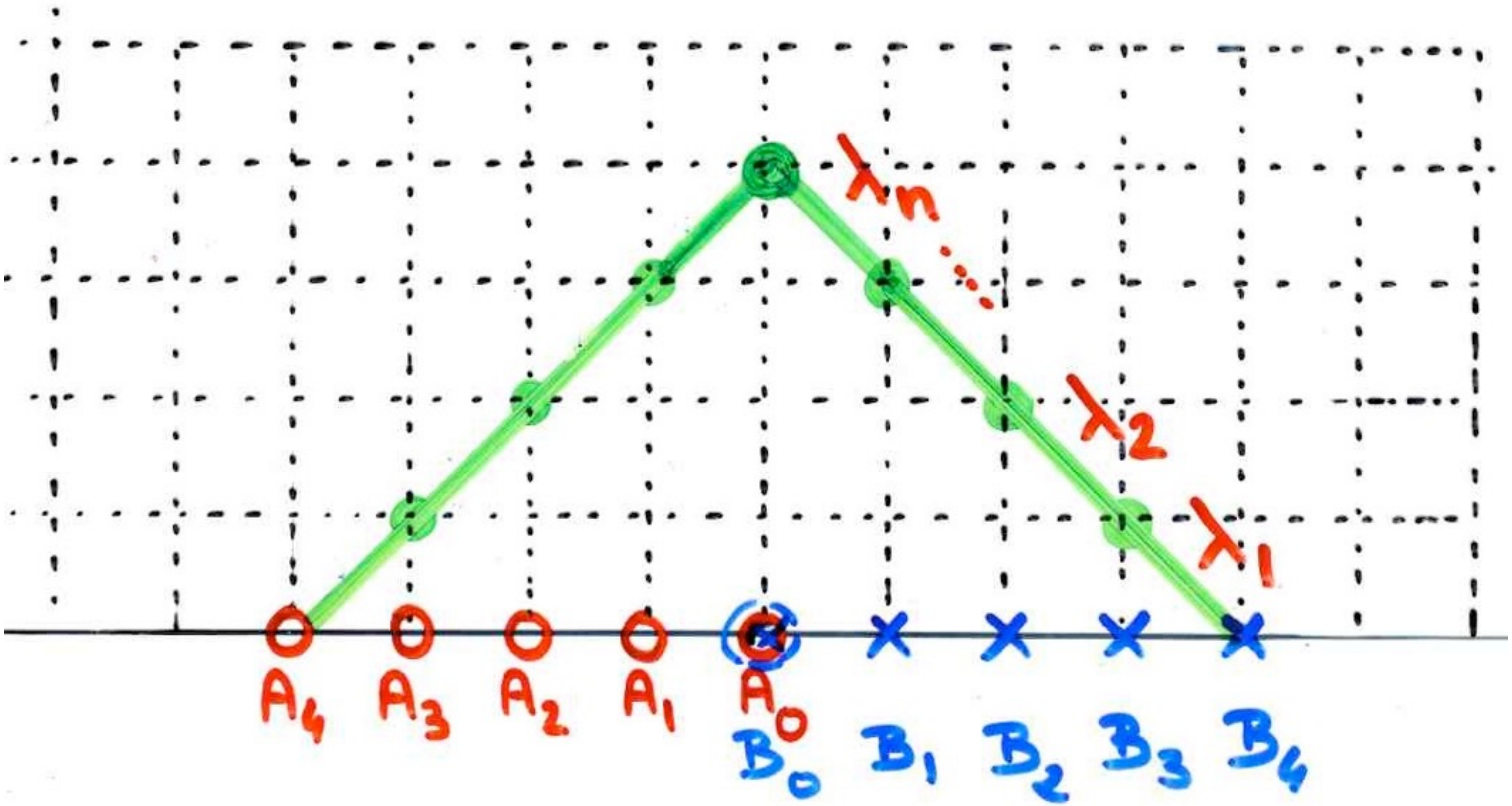
$$\Delta_n = H\begin{pmatrix} 0, & 1, & \dots, & n \\ 0, & 1, & \dots, & n \end{pmatrix}$$



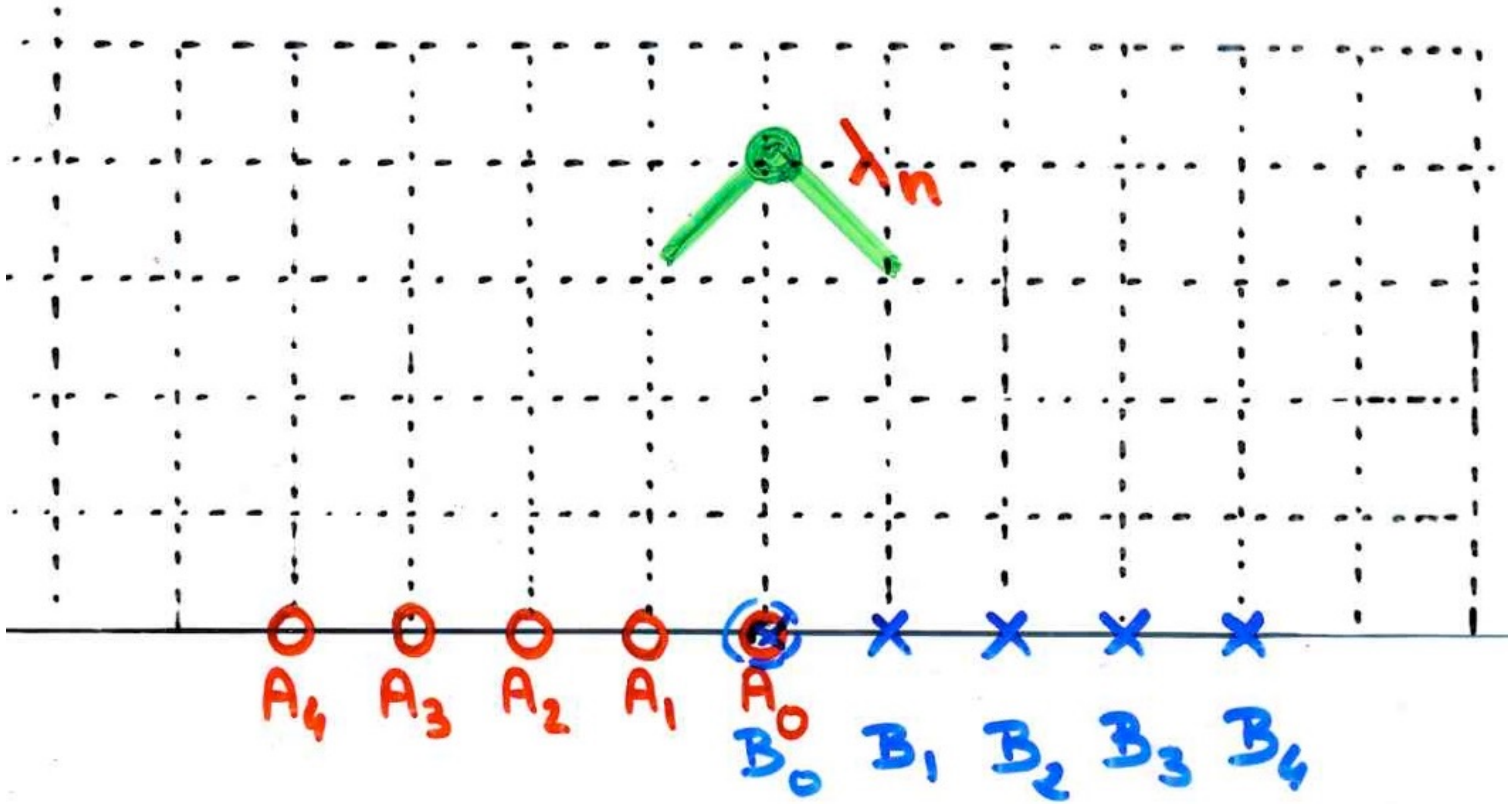


$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\Delta_n = H\begin{pmatrix} 0, & 1, & \ldots, & n \\ 0, & 1, & \ldots, & n \end{pmatrix}$$



$$\frac{\Delta_n}{\Delta_{n-1}}$$



$$\frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}} = \lambda_n$$

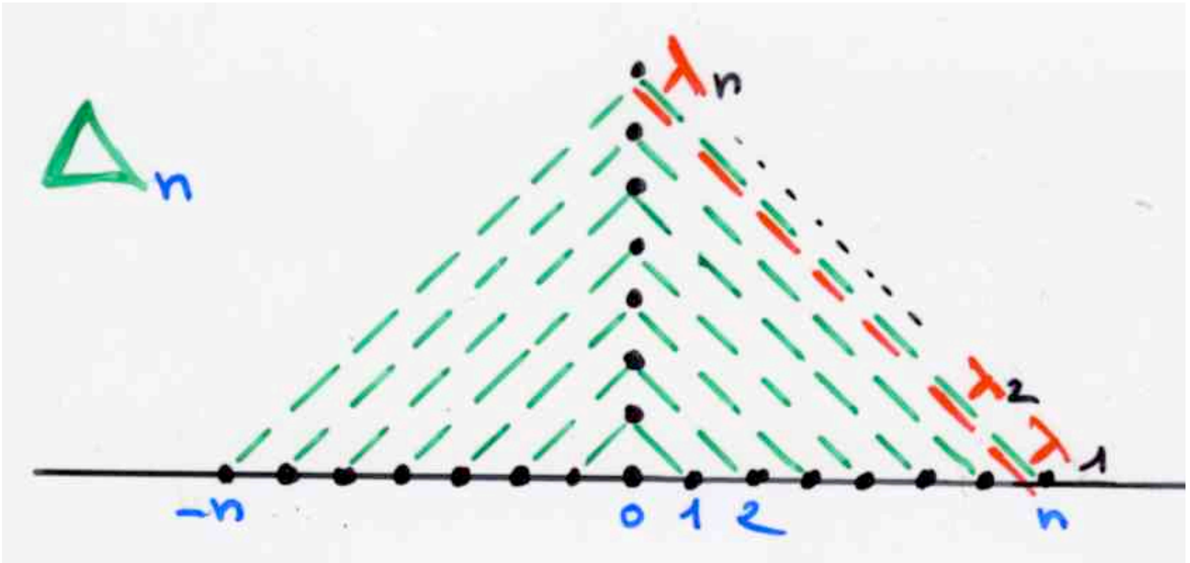
$\lambda_k \neq 0$, for every $k \geq 1$

$$\lambda_n = \frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}}$$

$$\lambda_n = \frac{\Delta_n \Delta_{n-2}}{\Delta^2_{n-1}}$$

$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\Delta_n = H\begin{pmatrix} 0, 1, \ldots, n \\ 0, 1, \ldots, n \end{pmatrix}$$



$$\Delta_n = (\lambda_1)^n (\lambda_2)^{n-1} \cdots (\lambda_{n-1})^2 \lambda_n$$

Proposition $\mathbb{K}$ field $\{\mu_n\}_{n\geq 0}$

there exist orthogonal polynomials having $\{\mu_n\}_{n\geq 0}$ as moments iff

$$\Delta_n \neq 0, \text{ for every } n\geq 0$$

in other words there exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ such that $\lambda_k \neq 0$

$$\mu_n = \sum_{\substack{|\omega|=n \\ \text{Motzkin path}}} v(\omega)$$

equivalenty

in other words there exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ $\lambda_k \neq 0$
such that

$$\sum_{n\geq 0} \mu_n t^n = J(t; b, \lambda)$$

Jacobi continued fraction

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

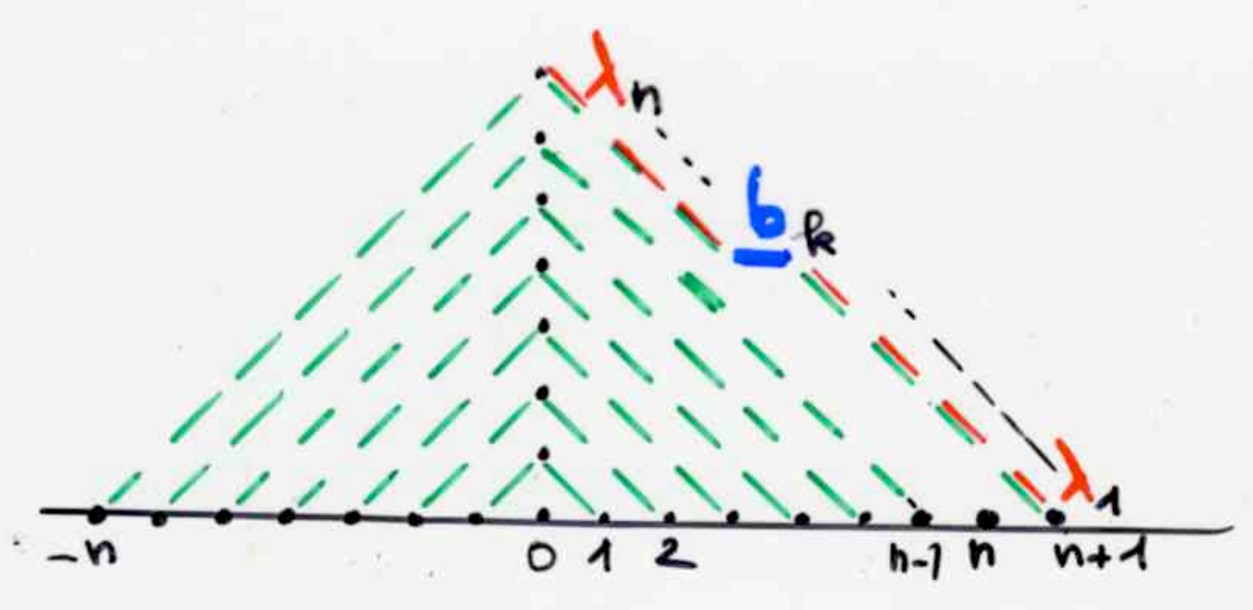
$J(t;b,\lambda)$

Jacobi continued fraction

$b=\{b_k\}_{k\geq 0}$ $\lambda=\{\lambda_k\}_{k\geq 1}$

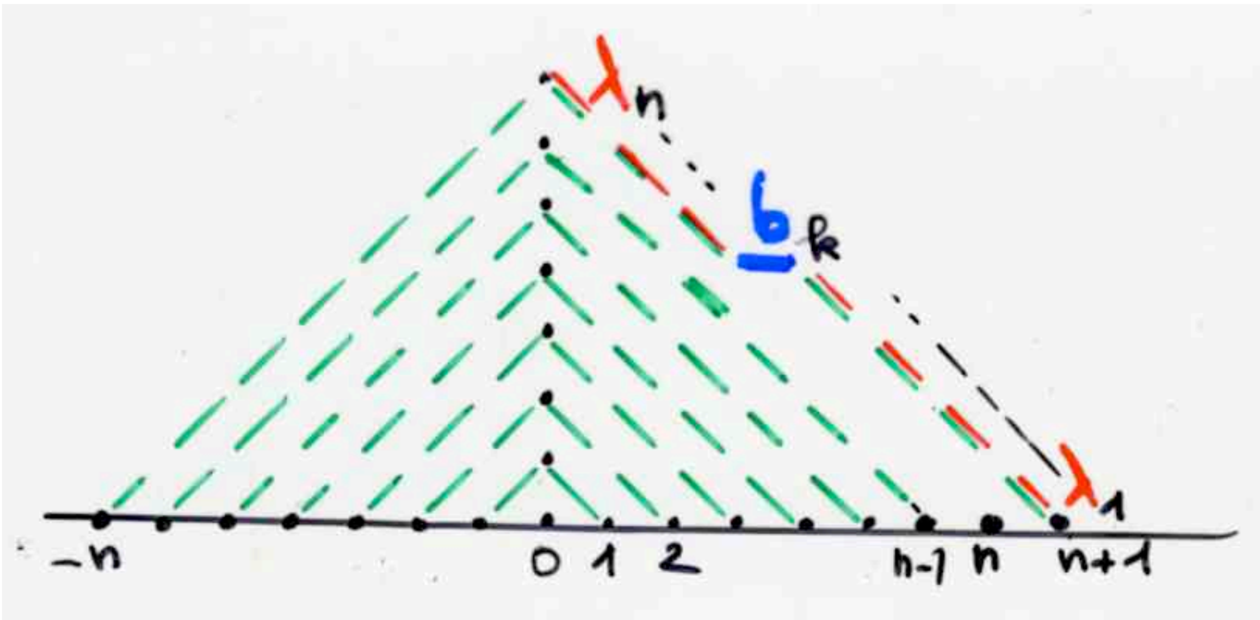
$$\chi_n = \det \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}$$

$$\chi_n = H\begin{pmatrix} 0, 1, \ldots, n-1; n \\ 0, 1, \ldots, n-1; n+1 \end{pmatrix}$$



$$\chi_n = (b_0 + \dots + b_n)\,\Delta_n$$

$$\chi_n = H\left(\begin{matrix}0, 1, \dots, n-1; n\\ 0, 1, \dots, n-1; n+1\end{matrix}\right)$$



$$x_n = (b_0 + \dots + b_n)\Delta_n$$

$$b_n = \frac{x_n}{\Delta_n} - \frac{x_{n-1}}{\Delta_{n-1}}$$

orthogonal polynomials

(or Stieljes continued fraction)

computing the coefficients

$$\lambda_k$$

with Hankel determinants of moments

continued fractions

Stieltjes

$$\underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}}_{S(t;\lambda)}$$

$$\mu_{2n+1} = 0$$
$$\mu_{2n} = \nu_n$$

$b_k = 0$ for every $k \geq 0$

$\{\nu_n\}_{n\geq 0}$

$$P_n(-x) = (-1)^n P_n(x)$$

for $n \geq 0$

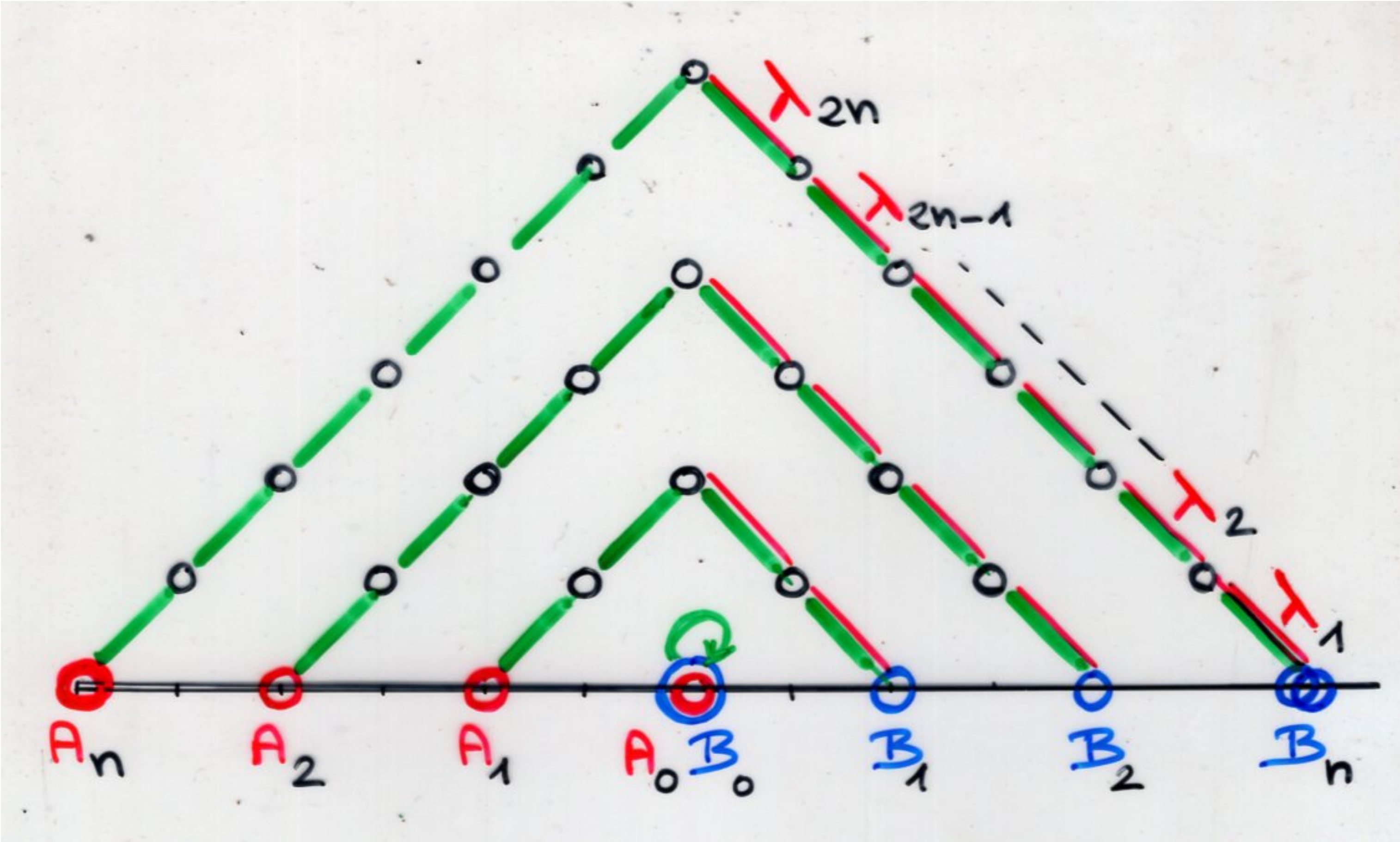
$$\sum_{|\omega|=2n} v(\omega) = \mu_n$$

Dyck path

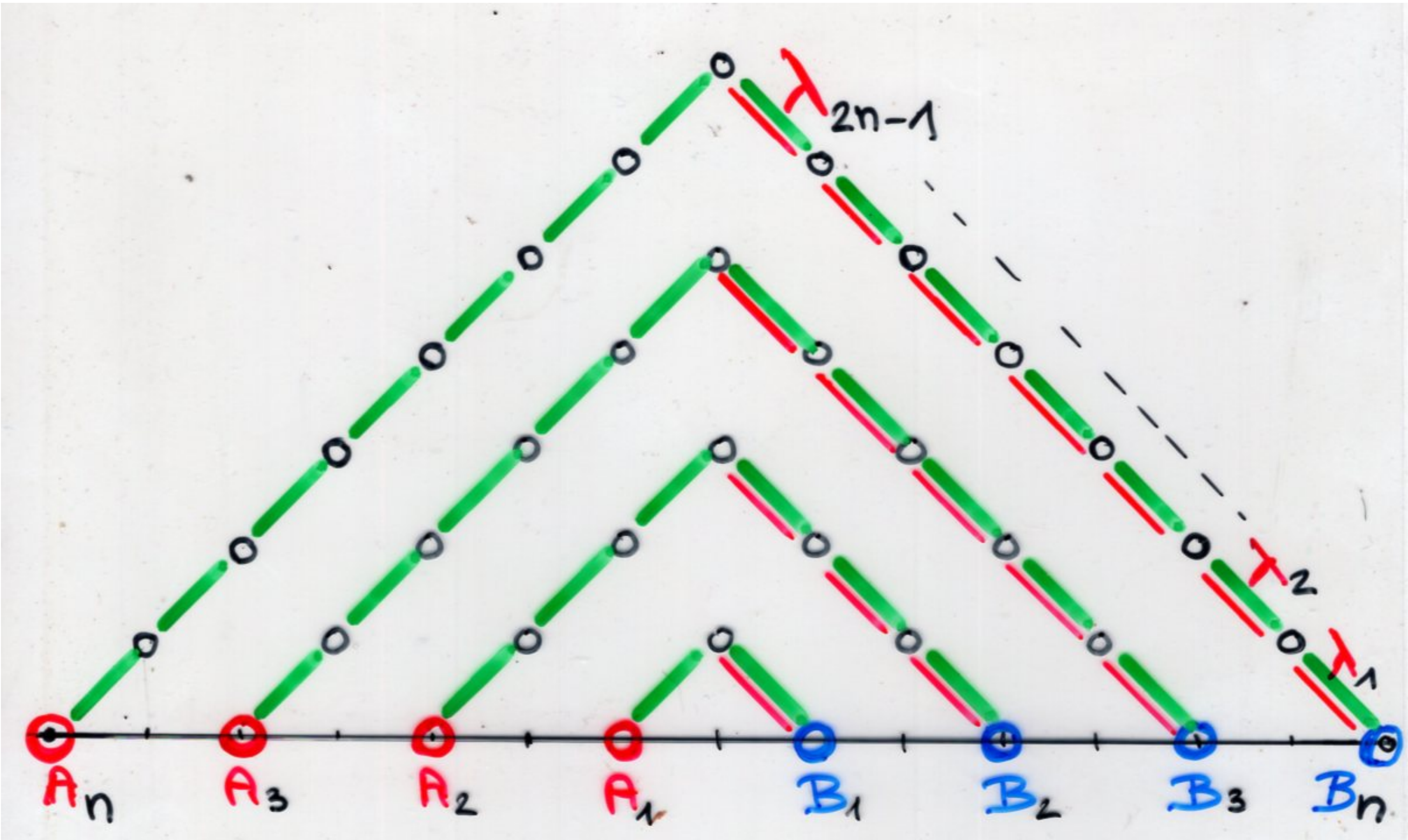
$$\Delta_n^{(0)}(\mu) = H_\mu\begin{pmatrix} 0, 1, \ldots, n \\ 0, 1, \ldots, n \end{pmatrix}$$

$$\Delta_n^{(1)}(\mu) = H_\mu\begin{pmatrix} 1, \ldots, n \\ 1, \ldots, n \end{pmatrix}$$

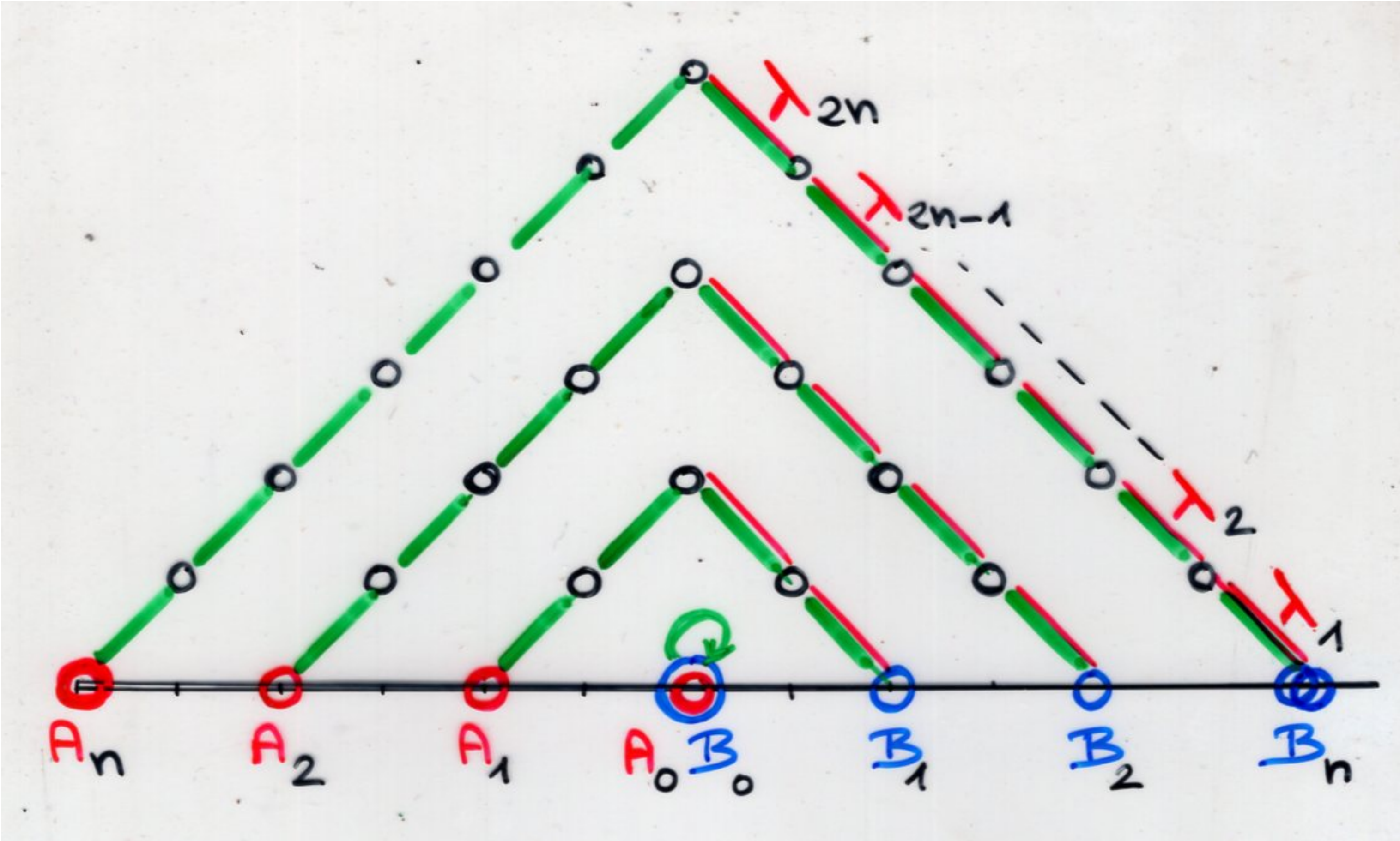
$$\Delta_n^{(0)}(\nu) = H_\nu\begin{pmatrix} 0, 1, \ldots, n \\ 0, 1, \ldots, n \end{pmatrix}$$

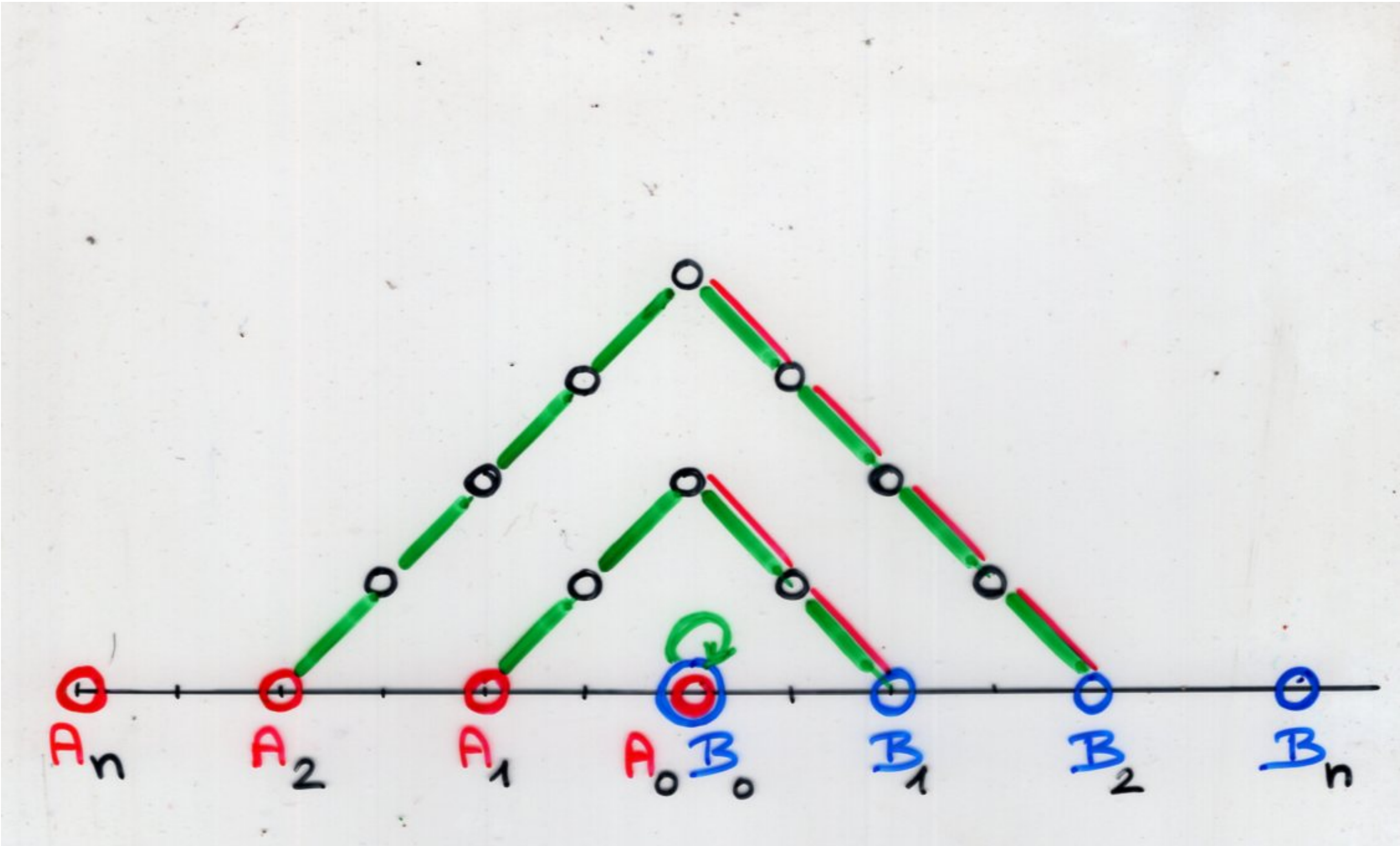


$$\Delta_n^{(1)}(\nu) = H_\nu\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix}$$

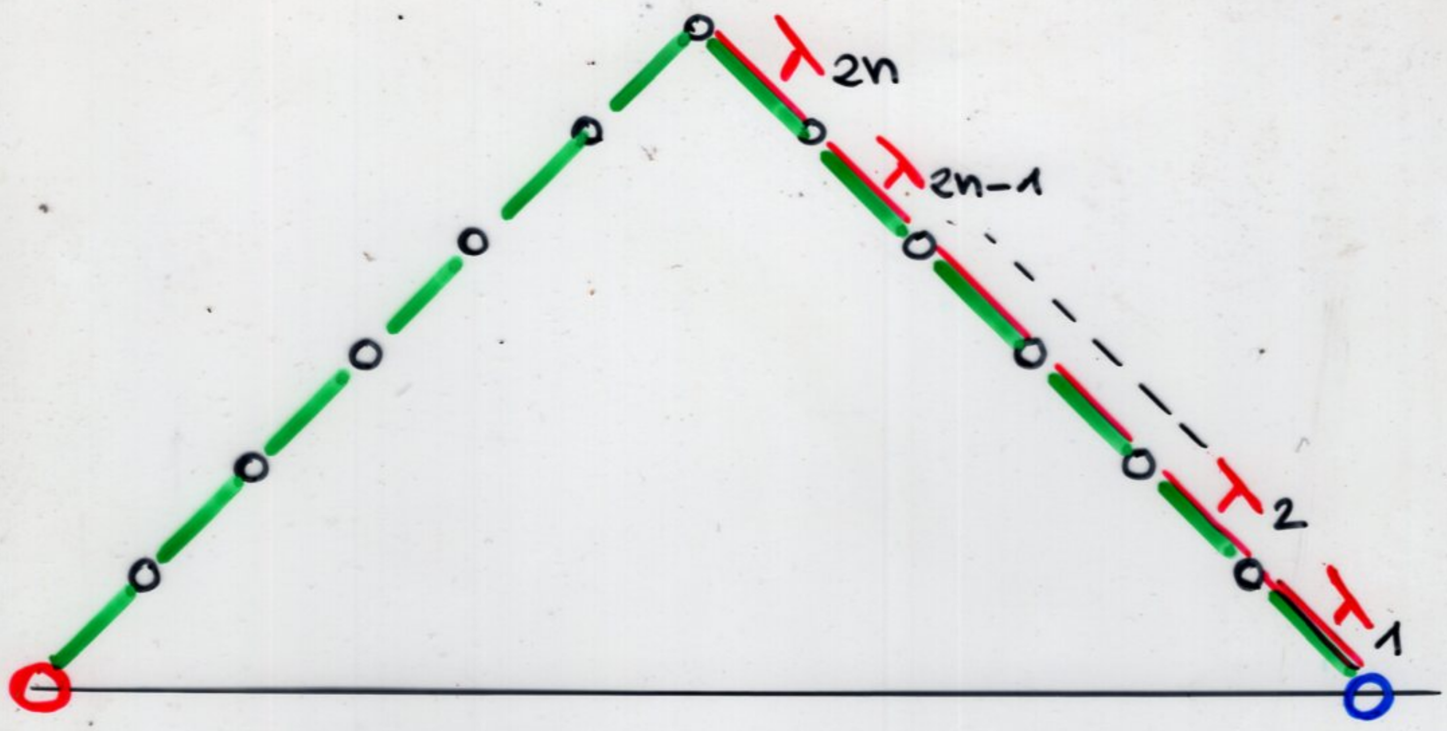


$$\Delta_n^{(0)}(\nu) = H_\nu\begin{pmatrix} 0, 1, \dots, n \\ 0, 1, \dots, n \end{pmatrix}$$



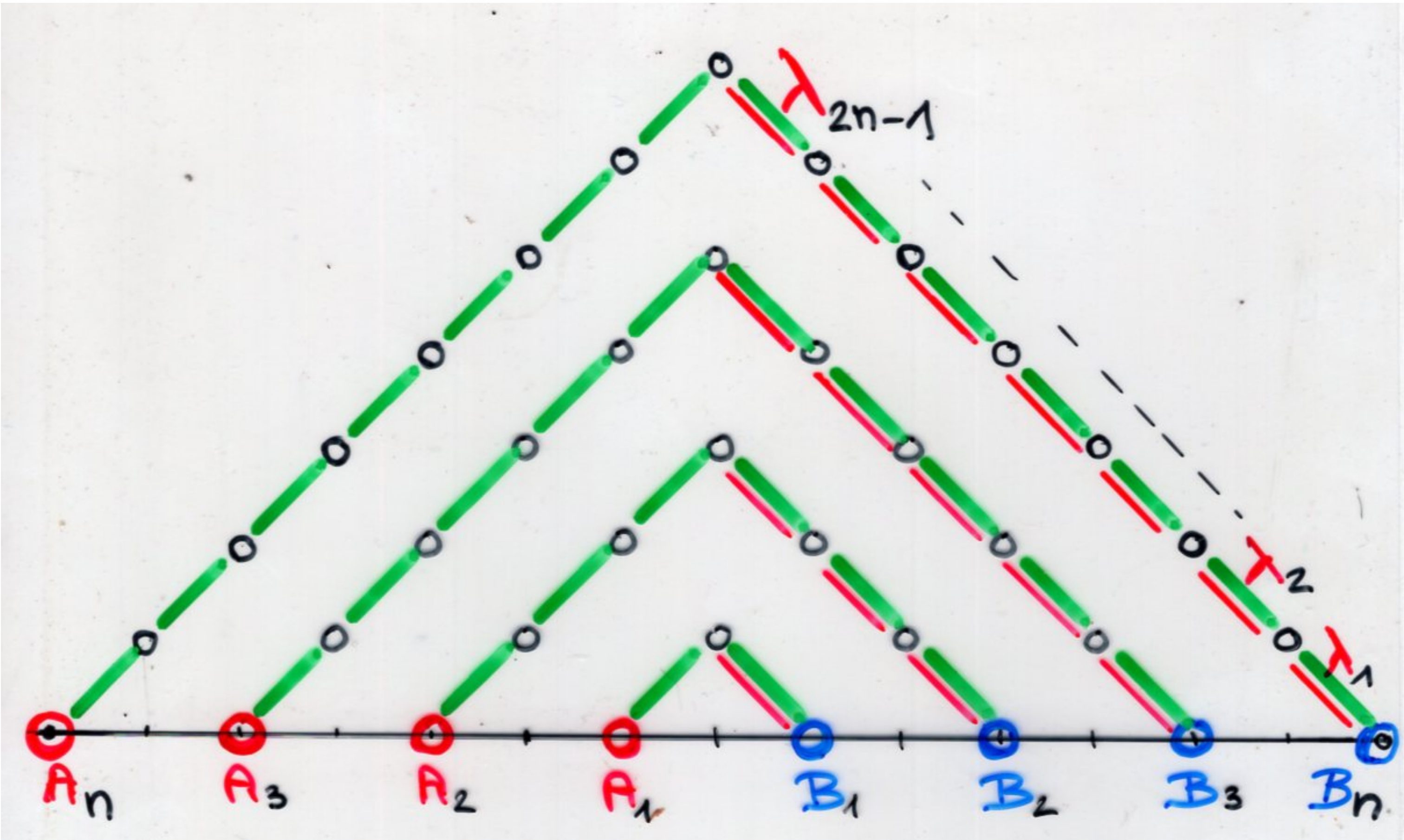


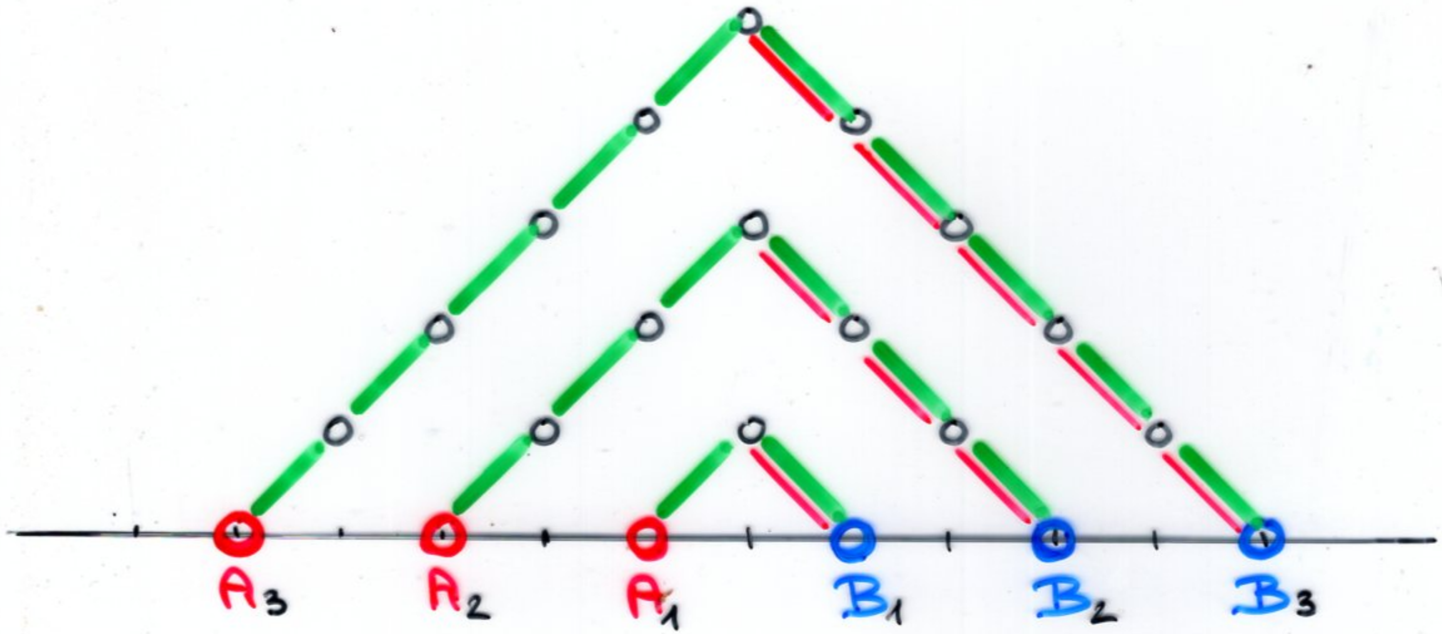
A_n
A_2
A_1
A_0 B_0
B_1
B_2
B_n



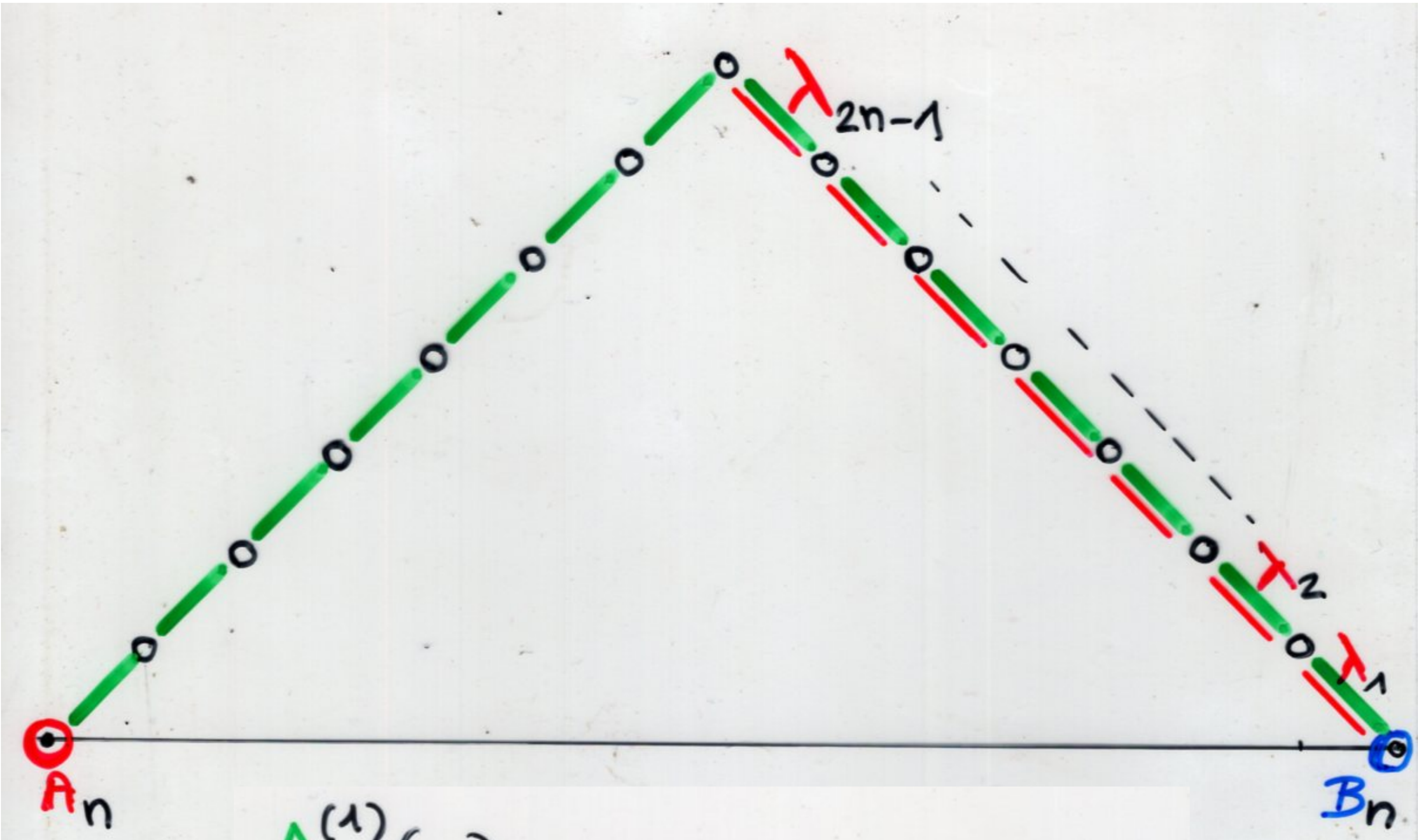
$$\frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} = \lambda_1 \lambda_2 \cdots \lambda_{2n}$$

$$\Delta_n^{(1)}(\nu) = H_\nu\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix}$$





A3
A2
A1
B1
B2
B3



$$\frac{\Delta_n^{(1)}(\nu)}{\Delta_{n-1}^{(1)}(\nu)} = \lambda_1 \, \lambda_2 \cdots \lambda_{2n-1}$$

$$\frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} = \lambda_1 \lambda_2 \cdots \lambda_{2n}$$

$$\frac{\Delta_n^{(1)}(\nu)}{\Delta_{n-1}^{(1)}(\nu)} = \lambda_1 \lambda_2 \cdots \lambda_{2n-1}$$

$$\lambda_{2n} = \frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} : \frac{\Delta_n^{(1)}(\nu)}{\Delta_{n-1}^{(1)}(\nu)} \quad (n \geq 1)$$

$$\lambda_{2n+1} = \frac{\Delta_{n+1}^{(1)}(\nu)}{\Delta_n^{(1)}(\nu)} : \frac{\Delta_n^{(0)}(\nu)}{\Delta_{n-1}^{(0)}(\nu)} \quad (n \geq 0)$$

$$\Delta_n^{(0)}(\nu) = H_\nu\begin{pmatrix} 0, 1, \dots, n \\ 0, 1, \dots, n \end{pmatrix}$$

$$\Delta_n^{(1)}(\nu) = H_\nu\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix}$$

$$\Delta_n^{(0)}(\nu) = (\lambda_1 \lambda_2)^n (\lambda_3 \lambda_4)^{n-1} \cdots (\lambda_{2n-1} \lambda_{2n})$$

$$\Delta_n^{(1)}(\nu) = \lambda_1^n (\lambda_2 \lambda_3)^{n-1} \cdots (\lambda_{2n-2} \lambda_{2n-1})$$

Corollary $\{\nu_n\}_{n\geq 0}$ $\nu_n \in \mathbb{K}$

There exist orthogonal polynomials with moments $\mu_{2n} = \nu_n$, $\mu_{2n+1} = 0$ iff

iff $\Delta_n^{(0)}(\nu) \neq 0$ and $\Delta_n^{(1)}(\nu) \neq 0$ for every $n \geq 0$

in other words there exist $\{\lambda_k\}_{k\geq 1}$ $\lambda_k \neq 0$ such that

$$\nu_n = \sum_{|\omega| = 2n} v(\omega)$$

Dyck paths

Corollary

$\{\nu_n\}_{n\geq 0}$ $\quad \nu_n \in \mathbb{K}$

in other words there exist $\{\lambda_k\}_{k\geq 1}$ $\lambda_k \neq 0$
such that

$$\sum_{n\geq 0} \nu_n t^n = S(t;\lambda)$$

Stieltjes continued fraction

iff $\quad \Delta_n^{(0)}(\nu) \neq 0$ and $\Delta_n^{(1)}(\nu) \neq 0$
for every $n \geq 0$

A classical determinant formula for orthogonal polynomials

Proposition

the ring $\mathbb{K}$ is a field.

Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of orthogonal polynomials with moments $\{\mu_n\}_{n\geq 0}$

Then $$P_n(x) = \frac{1}{\Delta_n} D_n(x)$$

where

$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

$$\Delta_n = H\binom{0, 1, \ldots, n}{0, 1, \ldots, n}$$

$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\{\mu_n\}_{n\geq 0}$$

$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

$$0 \leq p \leq n$$

$a_{n,p}$ = coefficient of x^p in $D_n(x)$

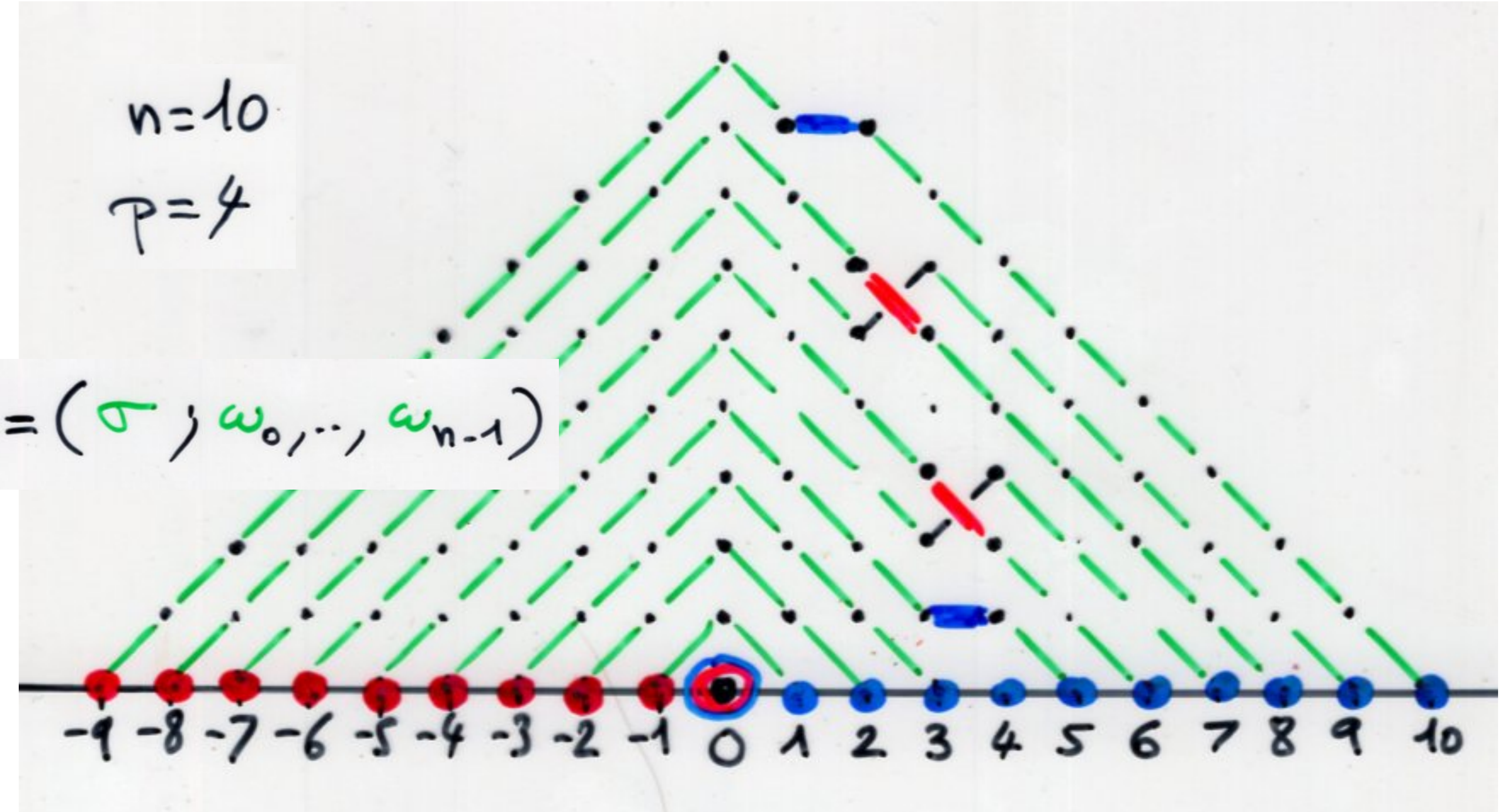
$$a_{n,p} = (-1)^{n-p}\, H\begin{pmatrix} 0, 1, \ldots\ldots\ldots\ldots, n-1 \\ 0, 1, \ldots, p-1, p+1, \ldots, n \end{pmatrix}$$

$a_{n,p} = (-1)^{n-p}$ $\quad H\begin{pmatrix} 0, & 1, & \ldots & \ldots & \ldots, & n-1 \\ 0, & 1, \ldots, & p-1, & p+1, & \ldots, & n \end{pmatrix}$

$n=10$

$p=4$

$\zeta = (\sigma \,;\, \omega_0, \ldots, \omega_{n-1})$



$A_i = (-i, 0)$ for $0 \leq i \leq n-1$

$$\begin{cases} B_i = (i, 0) & \text{for } 0 \leq i \leq p \\ B_i = (i+1, 0) & \text{for } p \leq i \leq n \end{cases}$$

$$H\begin{pmatrix} 0, 1, \dots\dots\dots\dots\dots, n-1 \\ 0, 1, \dots, p-1, p+1, \dots, n \end{pmatrix}$$

$$= \sum_{\zeta} (-1)^{inv(\sigma)} V(\omega_0) \cdots V(\omega_{n-1})$$

$$\zeta = (\sigma ; \omega_0, \dots, \omega_{n-1})$$

$\sigma \in \mathfrak{S}_n$ $\qquad$ $\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$

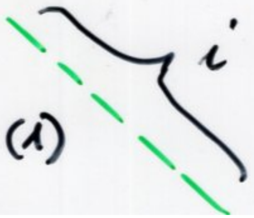
$\{\omega_i\}_{0 \leq i \leq n-1}$ $\qquad$ 2 by 2 disjoints

ω_i : i steps $\diagup$, followed by i steps $\diagdown$

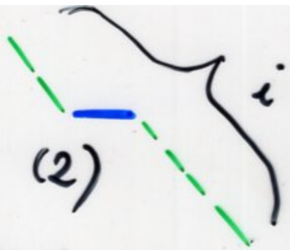
$(0 \leq i \leq p-1)$

for $p \leq i \leq n$, $\omega_i = \overbrace{\diagup \cdots \diagup}^{i} \times \omega_i'$

$\omega_i' =$

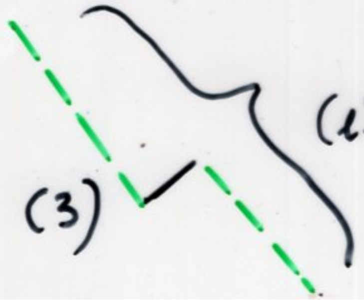


(1) i steps $\diagdown$



(2) i steps $\diagdown$

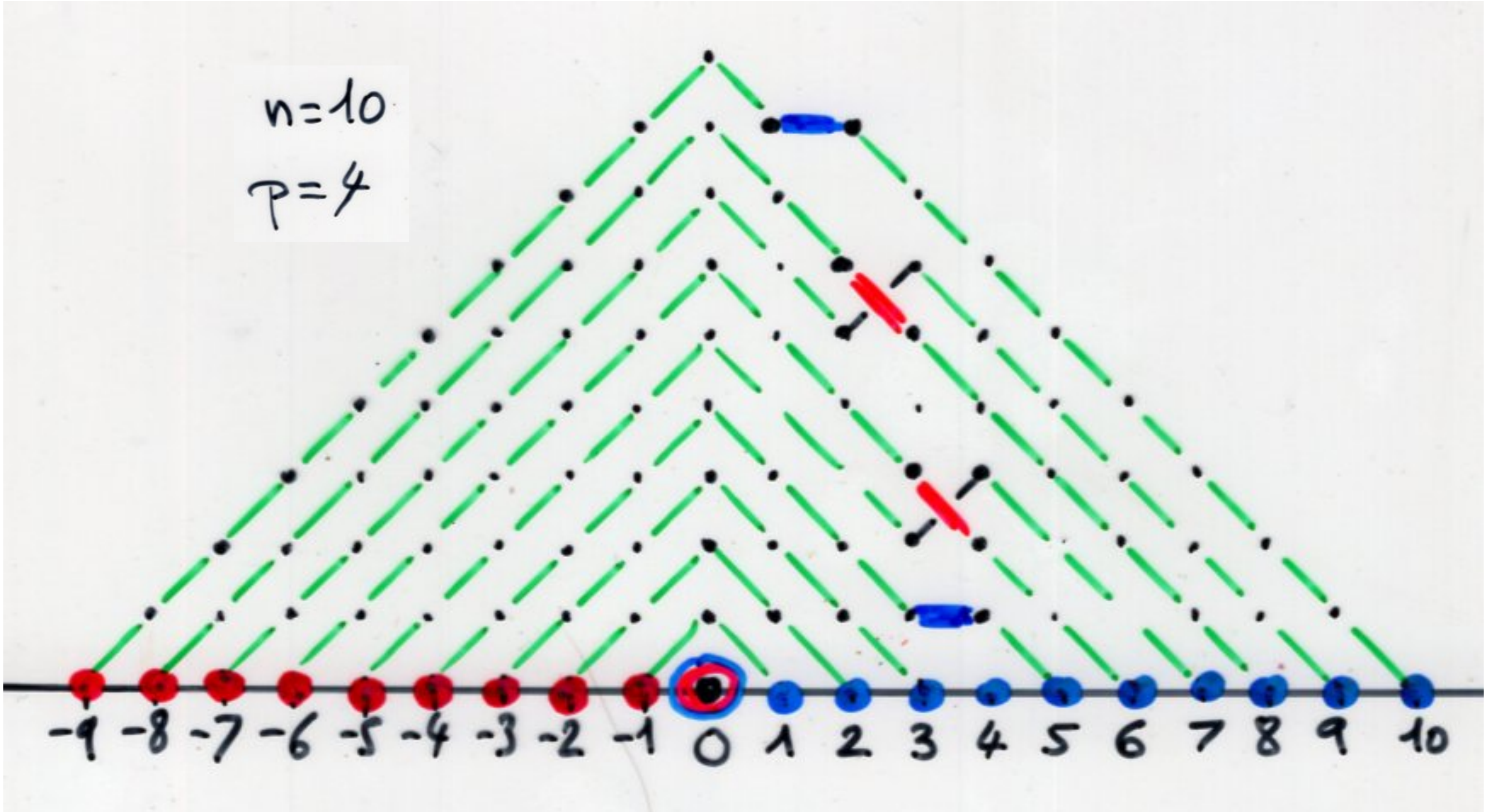
1 step —



(3) $(i+1)$ steps $\diagdown$

1 step $\diagup$

if ω_i type (3), then ω_{i+1} has type (1)


n=10
p=4
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

virtual crossings :

only once on a path, and between two consecutive paths ω_i and ω_{i+1}

$\omega_i : A_i \rightsquigarrow B_{i+1}$

$\omega_{i+1} : A_{i+1} \rightsquigarrow B_i$

$\sigma \in \mathfrak{S}_n$

$\sigma : [0, n-1] \to [0, n-1]$

$\sigma(i) = i$ if $0 \leq i < p$ or ω_i type (2)

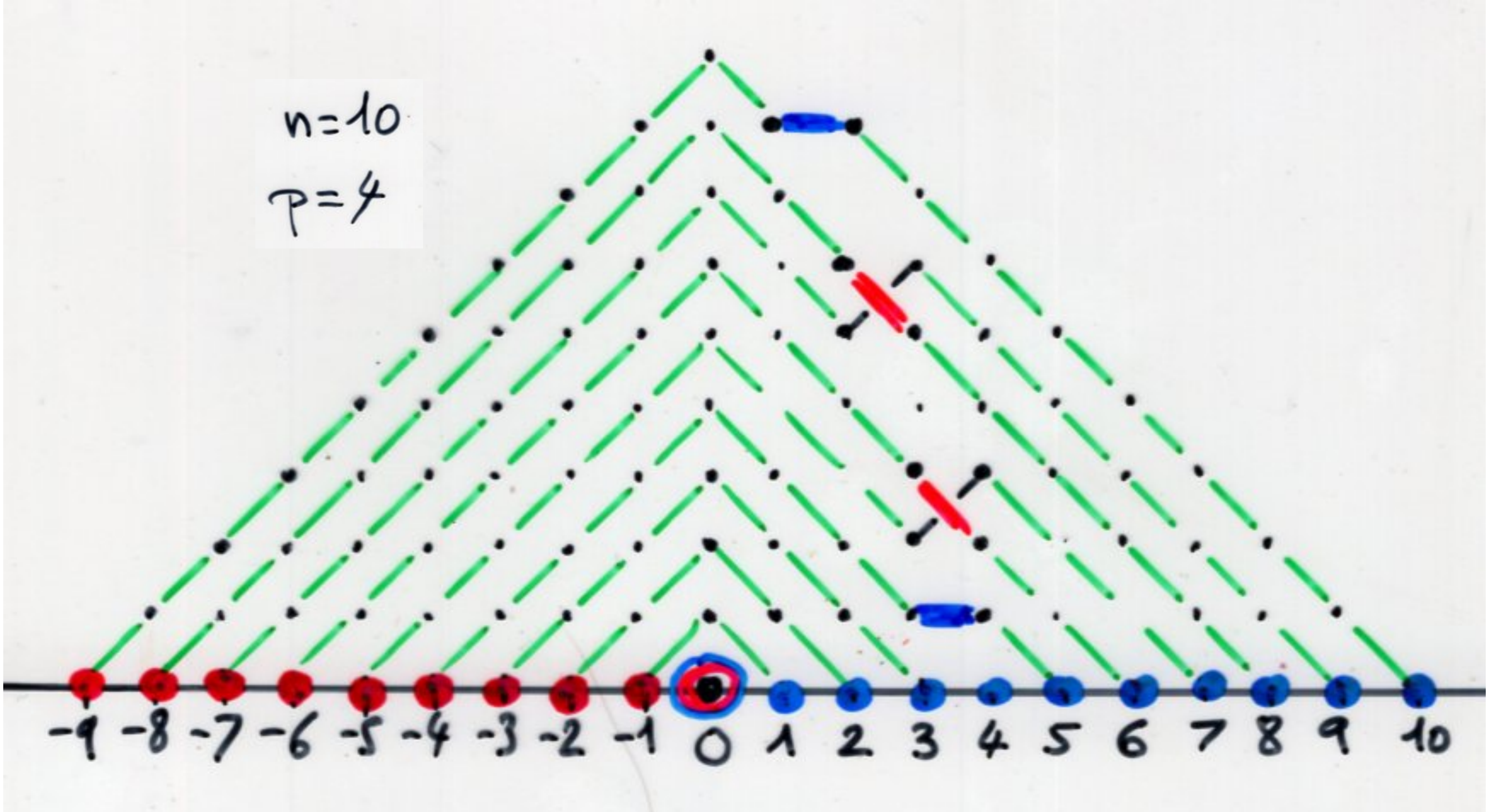
$\sigma(i) = i+1$ and $\sigma(i+1) = i$

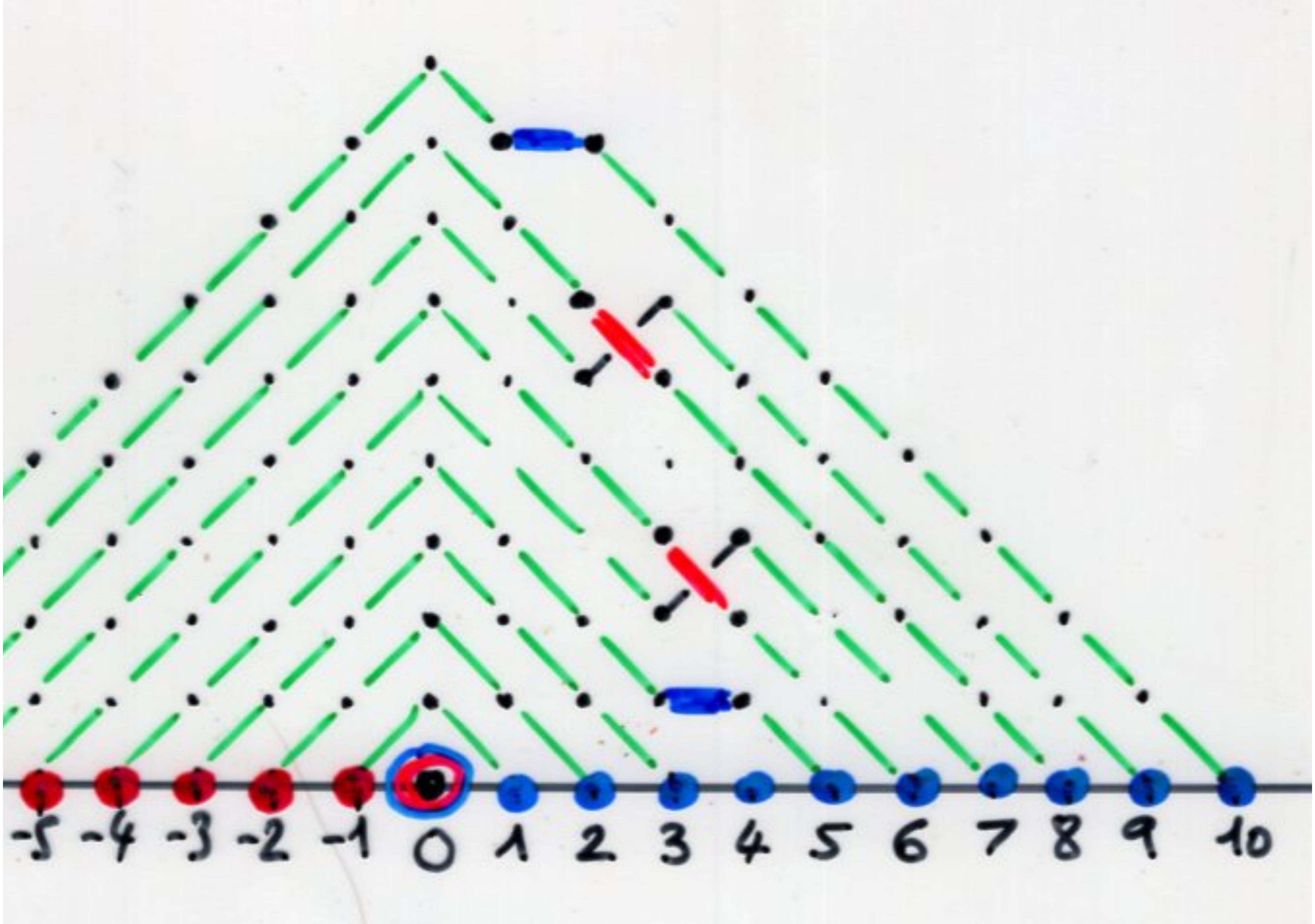
if ω_i type (3) (and thus ω_{i+1} type (1)) bijection:

$$\xi = (\sigma; \omega_0, \ldots, \omega_{n-1}) \longrightarrow \beta$$

pavage of $[0, n-1]$

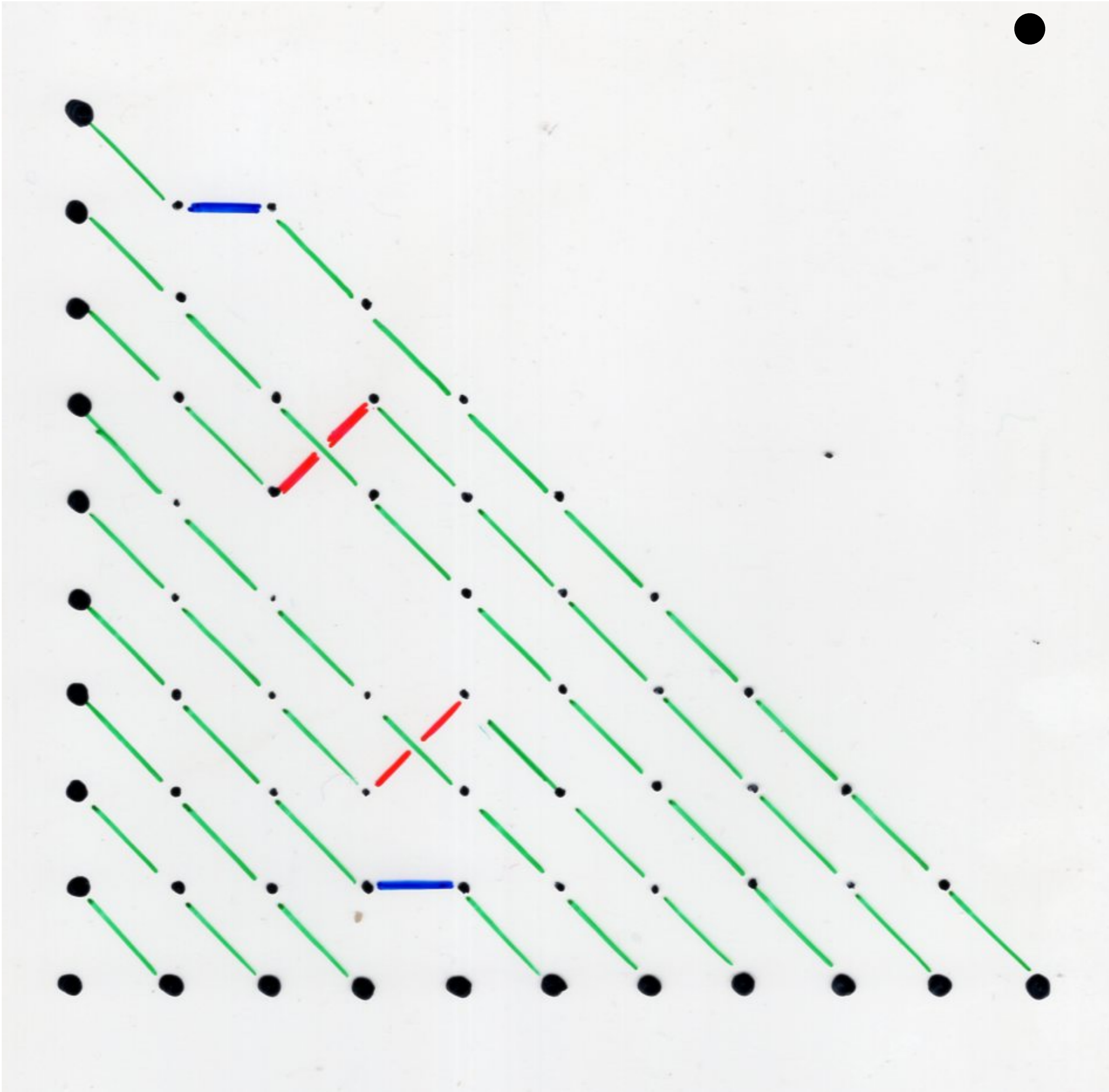
(or Favard path)


n=10
p=4
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10


-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

$n = 10$

$p = 4$



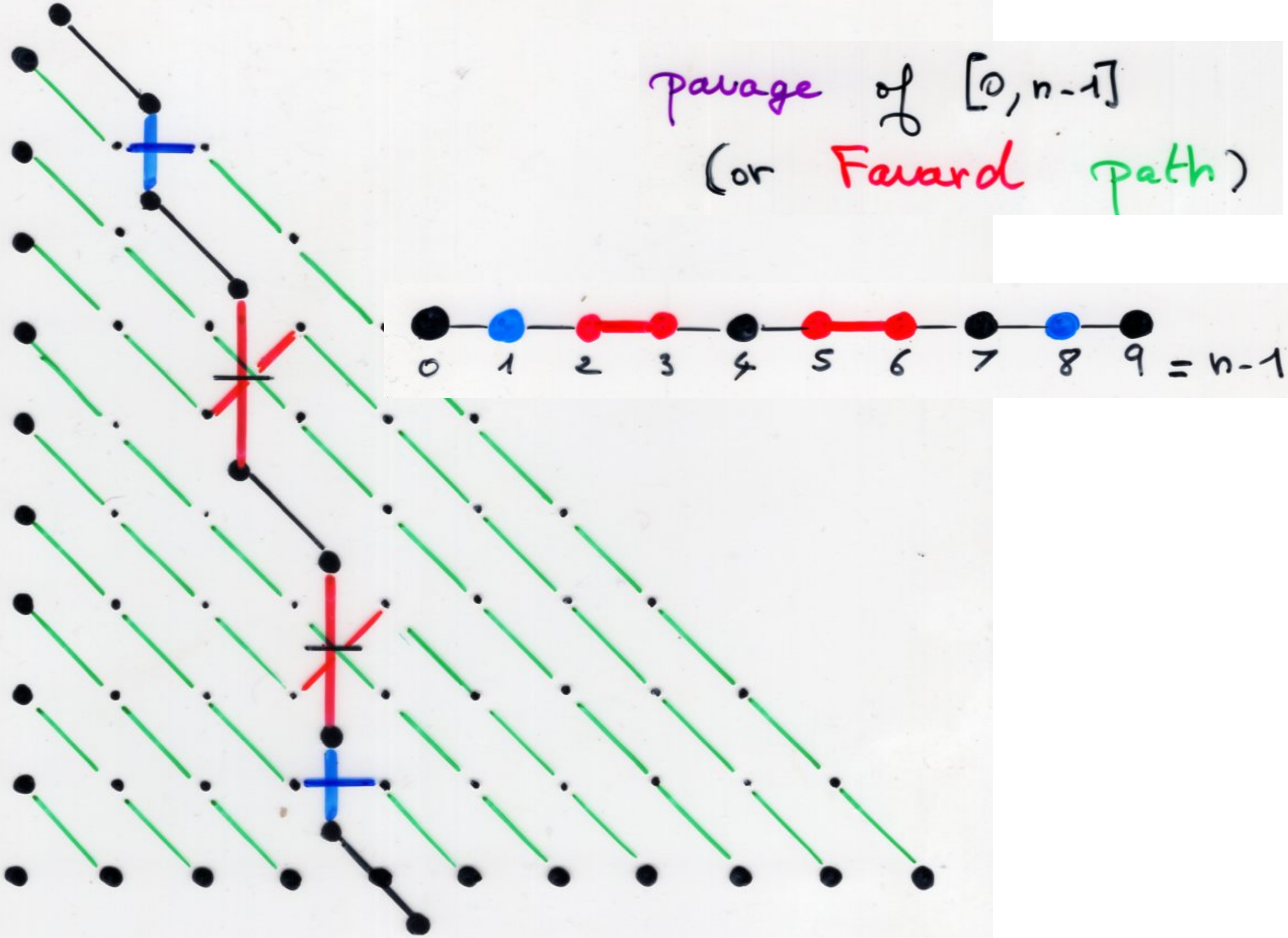
bijection: $\xi = (\sigma; \omega_0, .., \omega_{n-1}) \longrightarrow \beta$

$n = 10$

$p = 4$

pavage of $[0, n-1]$

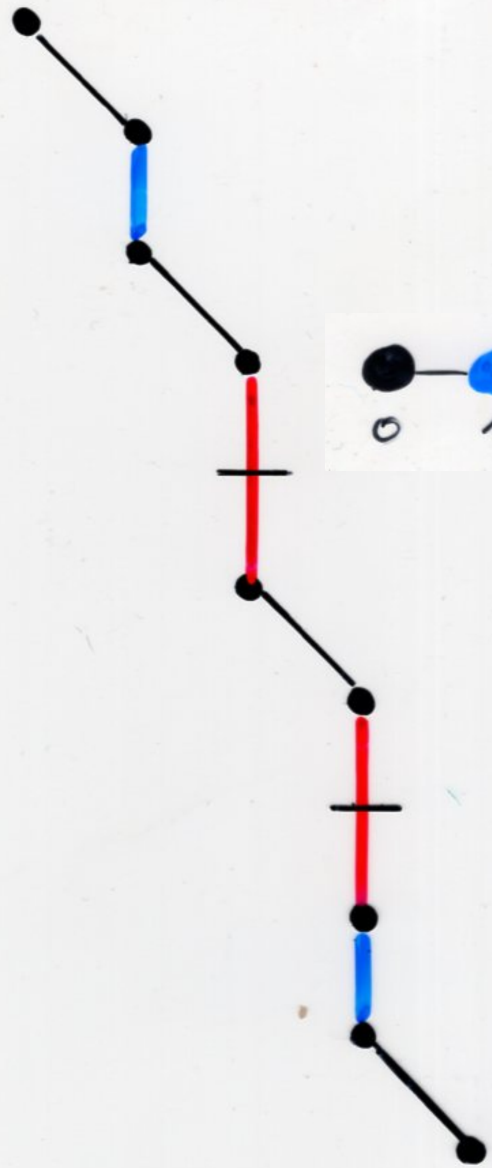
(or Favard path)



bijection: $\xi = (\sigma; \omega_0, \dots, \omega_{n-1}) \longrightarrow \beta$

$n = 10$

$p = 4$



pavage of $[0, n-1]$
(or Favard path)

here $v(\beta) = b_1 \, b_8 \, \lambda_3 \, \lambda_6$

bijection:

$\zeta = (\sigma; \omega_0, \dots, \omega_{n-1}) \longrightarrow \beta$

$n = 10$

$p = 4$

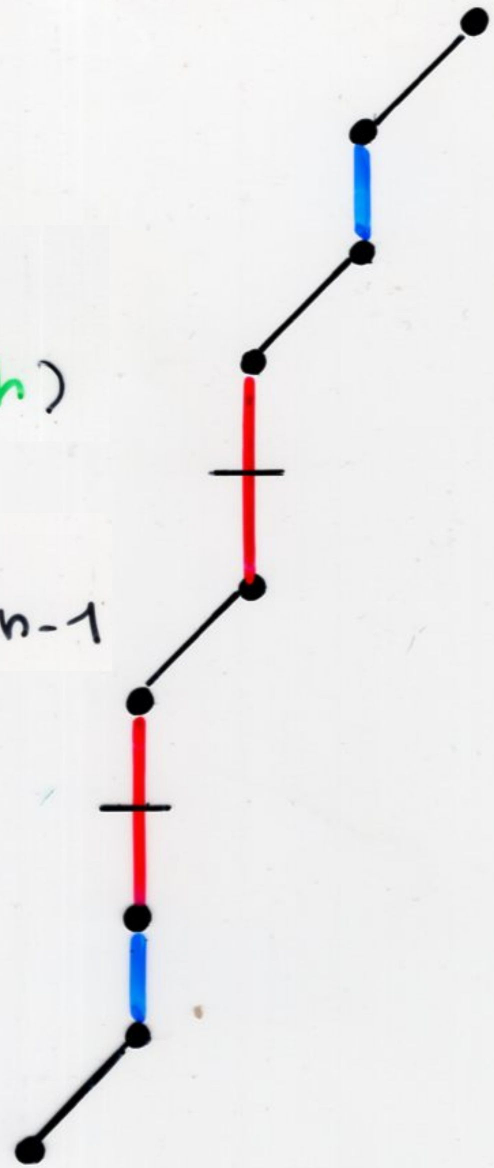
pavage of $[0, n-1]$

(or Favard path)

$= n-1$

here $v(\beta) = b_1 \, b_8 \, \lambda_3 \, \lambda_6$

pavage $\beta \longleftrightarrow$ Favard path η



$$H\left(\begin{matrix}0, & 1, & \dots\dots\dots\dots & & \dots, & n-1\\ 0, & 1, & \dots, \; p-1, & p+1, & \dots, & n\end{matrix}\right)$$

$$= \sum_{\zeta} (-1)^{\mathrm{inv}(\sigma)} V(\omega_0) \cdots V(\omega_{n-1})$$

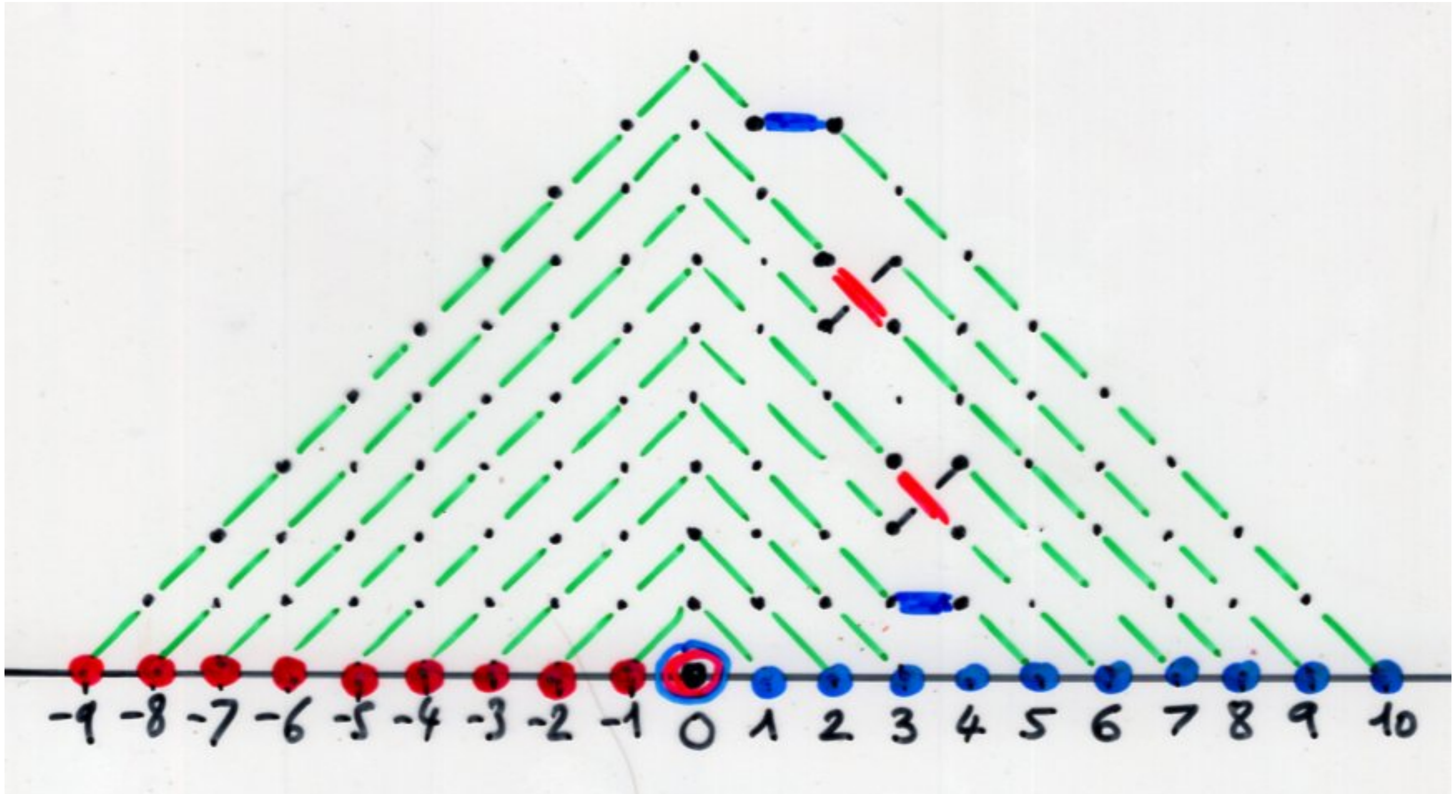
$$\zeta = (\sigma \,; \omega_0, \dots, \omega_{n-1})$$

$$\sigma \in \mathfrak{S}_n \qquad \omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

$$\{\omega_i\}_{0 \leq i \leq n-1}$$ 2 by 2 disjoints

$$V(\omega_0)\cdots V(\omega_{n-1}) = V(\beta)\,\Delta_n$$

here $\quad V(\beta) = b_1\, b_8\, \lambda_3\, \lambda_6$



$$v(\omega_0)\cdots v(\omega_{n-1}) = v(\beta)\,\Delta_n$$

$$H\begin{pmatrix} 0, & 1, & \cdots & \cdots & \cdots & \cdots, & n-1 \\ 0, & 1, & \cdots, & p-1, & p+1, & \cdots, & n \end{pmatrix} = \sum_{\sigma} (-1)^{inv(\sigma)}\, v(\omega_0)\cdots v(\omega_{n-1})$$

$$inv(\sigma) = d(\beta)$$

$d(\beta) =$ number of dimers of the pavage β (or number of NN steps of the Favard path η)

$$H\begin{pmatrix} 0, & 1, & \cdots & \cdots & \cdots & \cdots, & n-1 \\ 0, & 1, & \cdots, & p-1, & p+1, & \cdots, & n \end{pmatrix} = \sum_{\substack{\beta \\ \text{pavage of } [0,n-1]}} (-1)^{d(\beta)}\, v(\beta)\,\Delta_n$$

$$\mathbf{H}\begin{pmatrix} 0, & 1, & \cdots & \cdots & \cdots & , n-1 \\ 0, & 1, & \ldots, & p-1, & p+1, & \ldots, n \end{pmatrix} = \sum_{\substack{\beta \\ \text{pavage of } [0,n-1]}} (-1)^{d(\beta)} v(\beta) \Delta_n$$

$$ip(\beta) = p$$

(number of isolated points)

$$a_{n,p} = (-1)^{n-p} \; \mathbf{H}\begin{pmatrix} 0, & 1, & \cdots & \cdots & \cdots & , n-1 \\ 0, & 1, & \ldots, & p-1, & p+1, & \ldots, n \end{pmatrix}$$

$$n - p = m(\beta) + 2\, d(\beta)$$

$m(\beta)$ = number of monomers of the pavage β

$$(-1)^{n-p} = (-1)^{m(\beta)}$$

$$a_{n,p} = \sum_{\substack{\beta \\ \text{pavage of } [0,n-1]}} (-1)^{m(\beta)+d(\beta)} v(\beta)\, \Delta_n$$

$$\underbrace{\sum_{0 \leq p \leq n} a_{n,p}\, x^p}_{D_n(x)} = \underbrace{\sum_{\substack{\beta \\ \text{pavage of} \\ [0,n-1]}} (-1)^{m(\beta)+d(\beta)} v(\beta)\, x^{ip(\beta)}}_{P_n(x)}$$

Δ_n

$$P_n(x) = \frac{1}{\Delta_n} D_n(x)$$

where

$$\Delta_n = H\begin{pmatrix} 0, & 1, & \dots, & n \\ 0, & 1, & \dots, & n \end{pmatrix}$$

$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

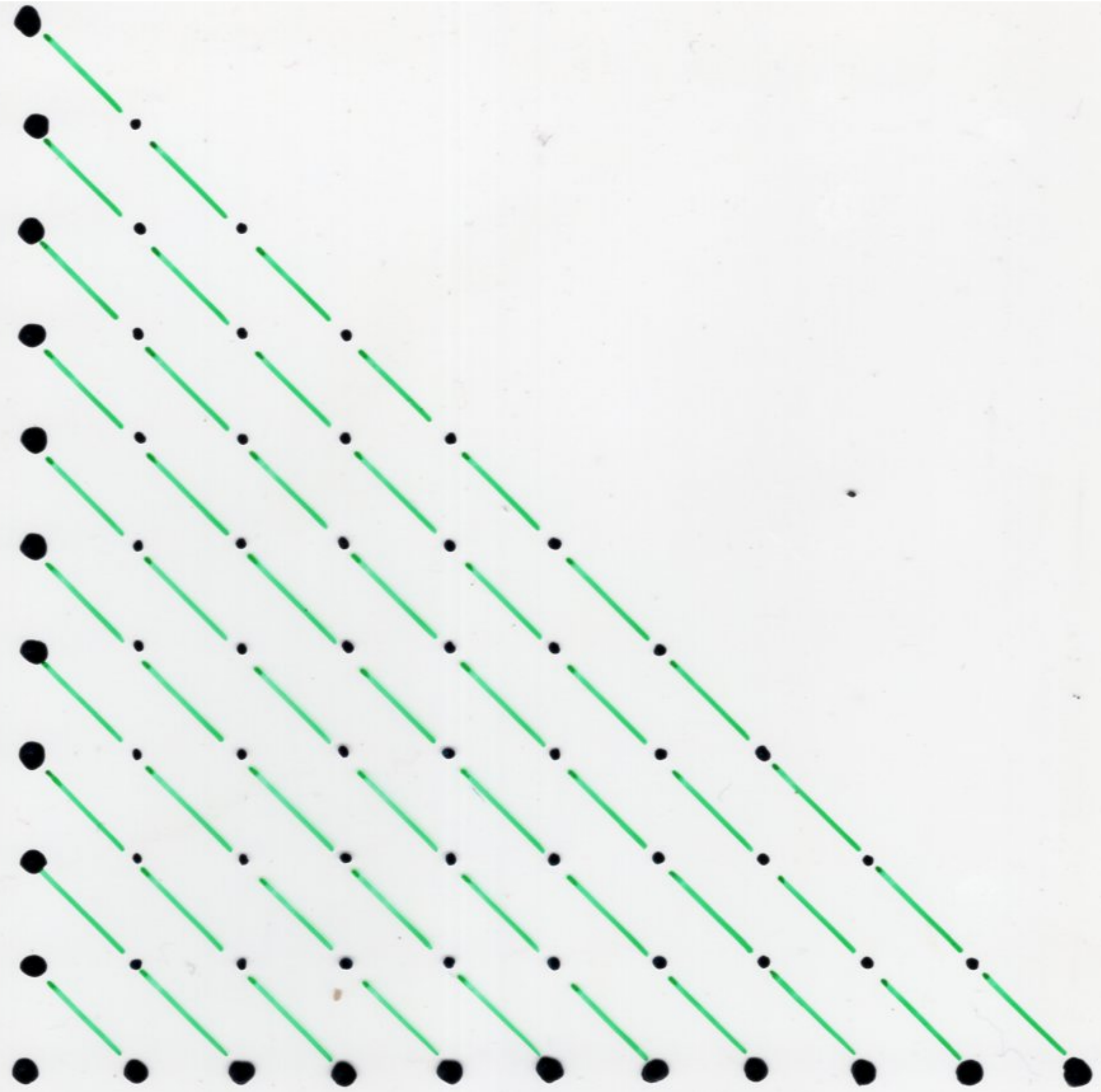
$$D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

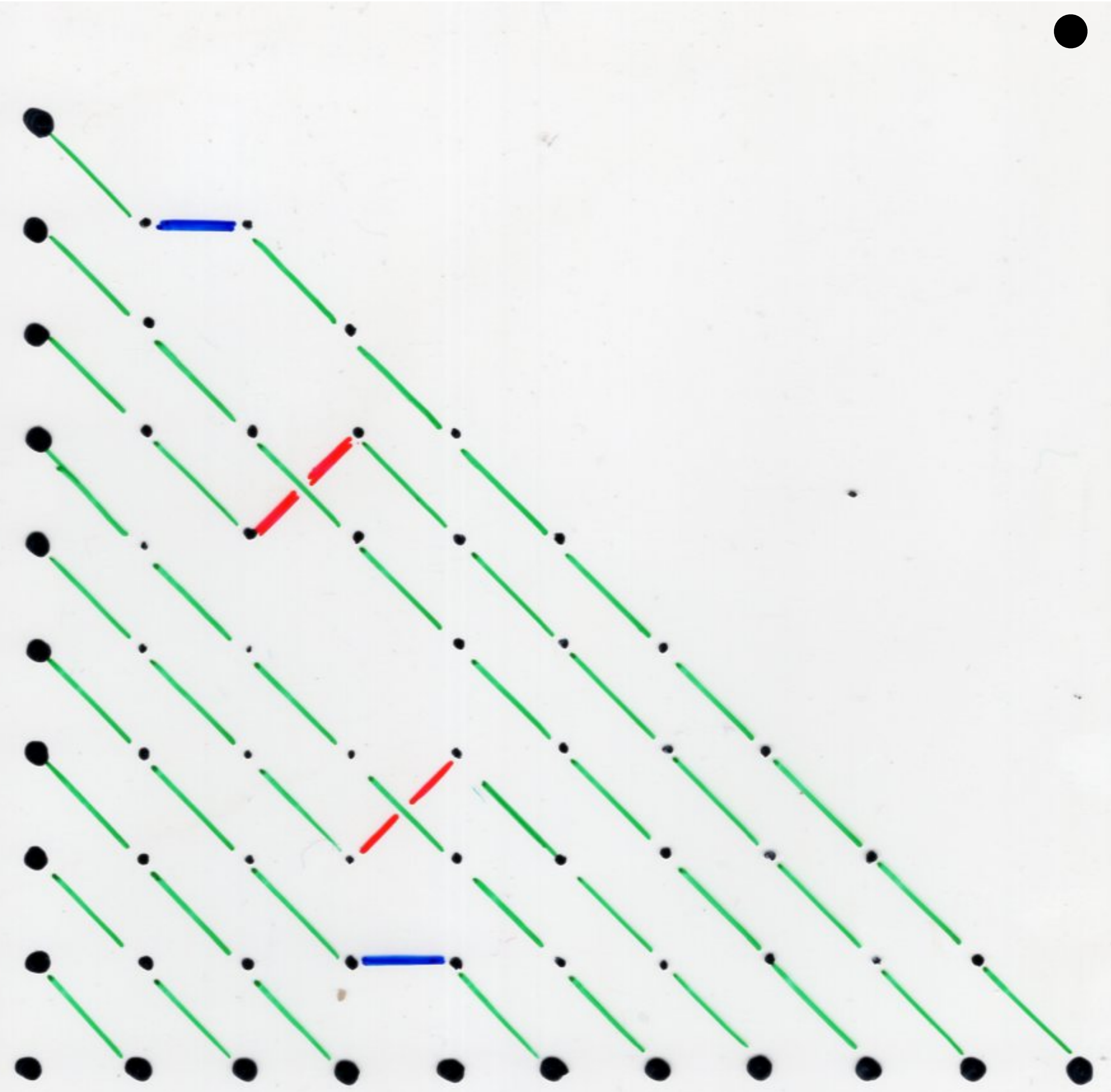
end of the proof $\square$

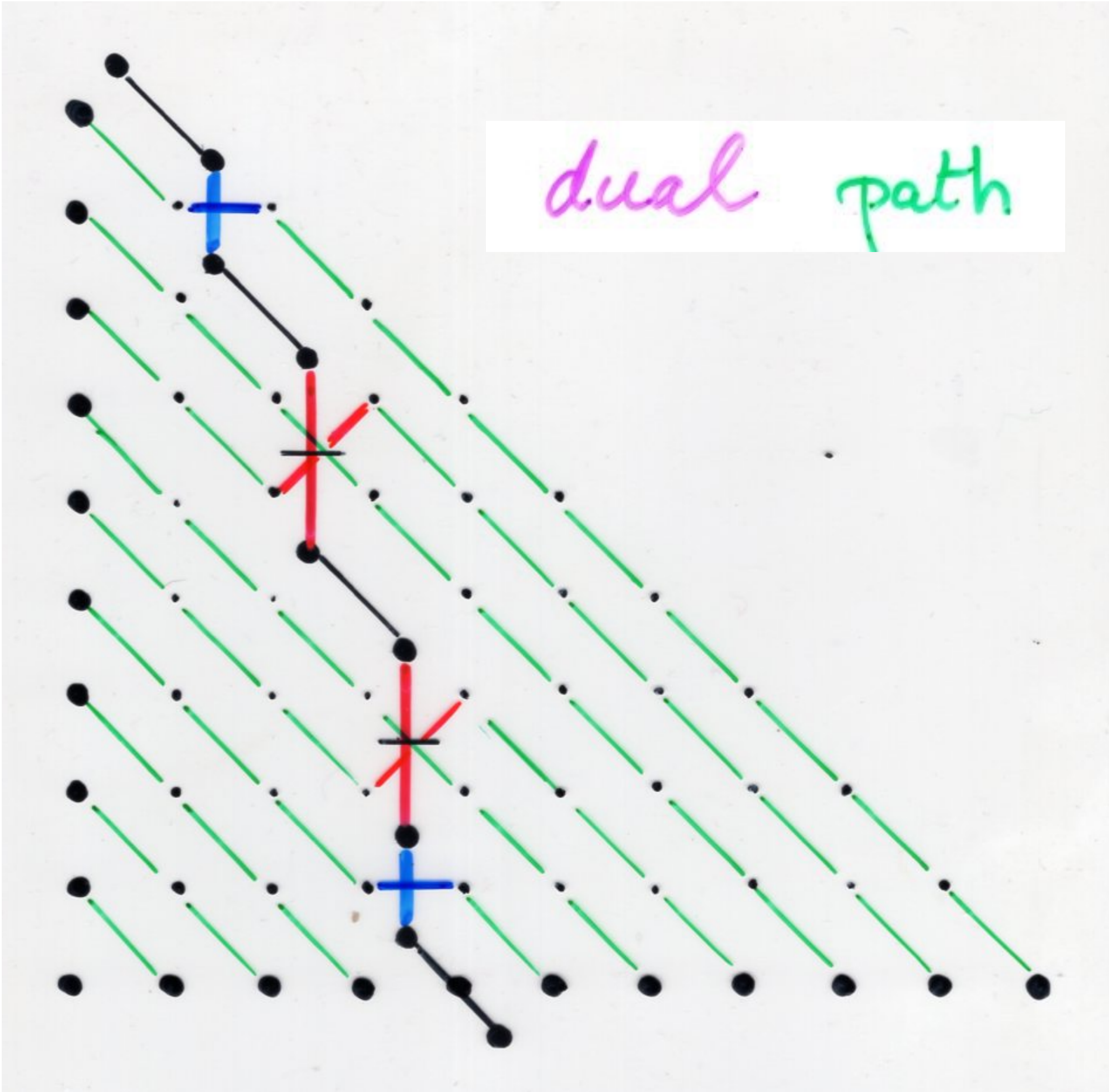
Duality

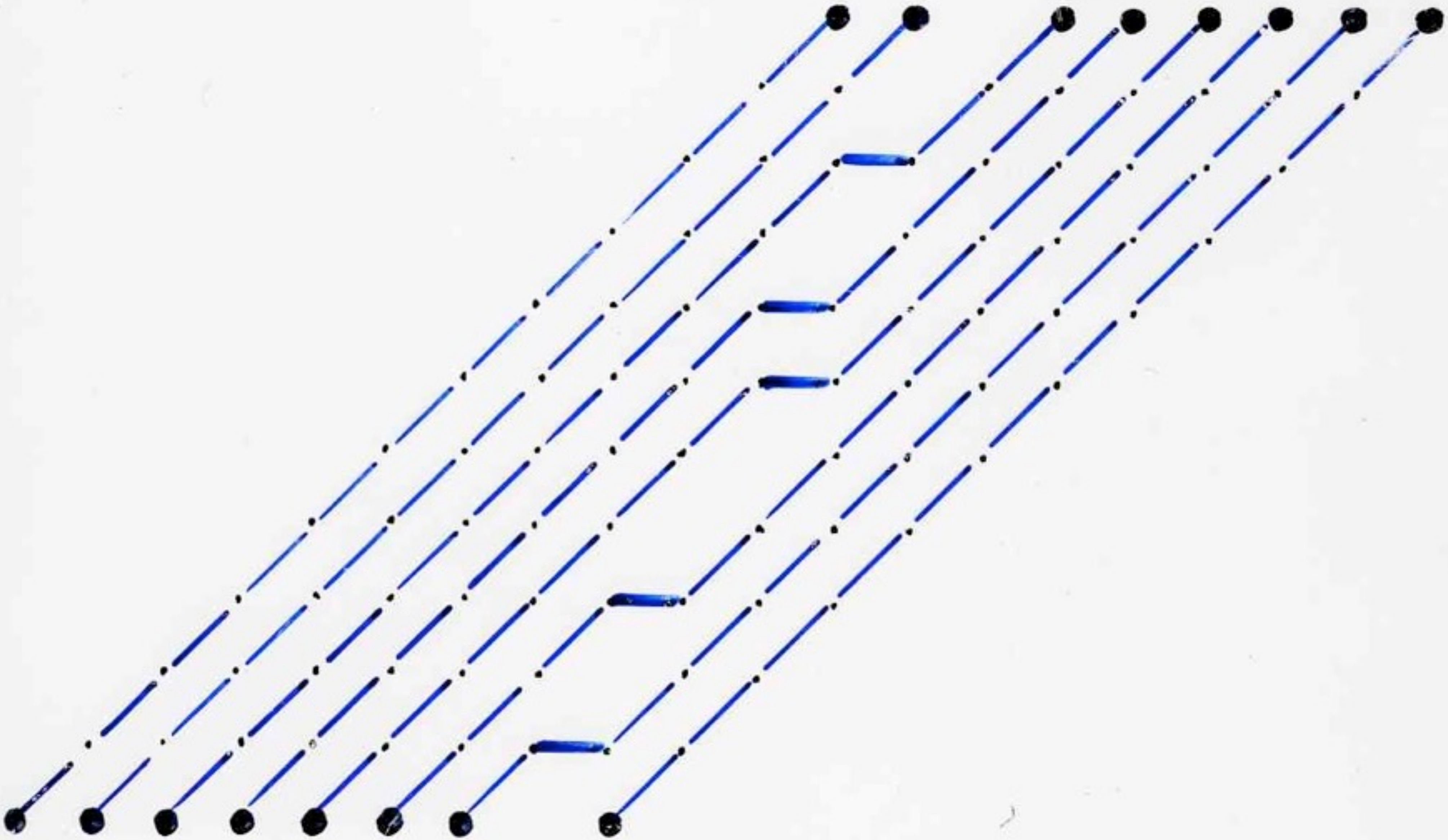
(the idea of duality in paths)

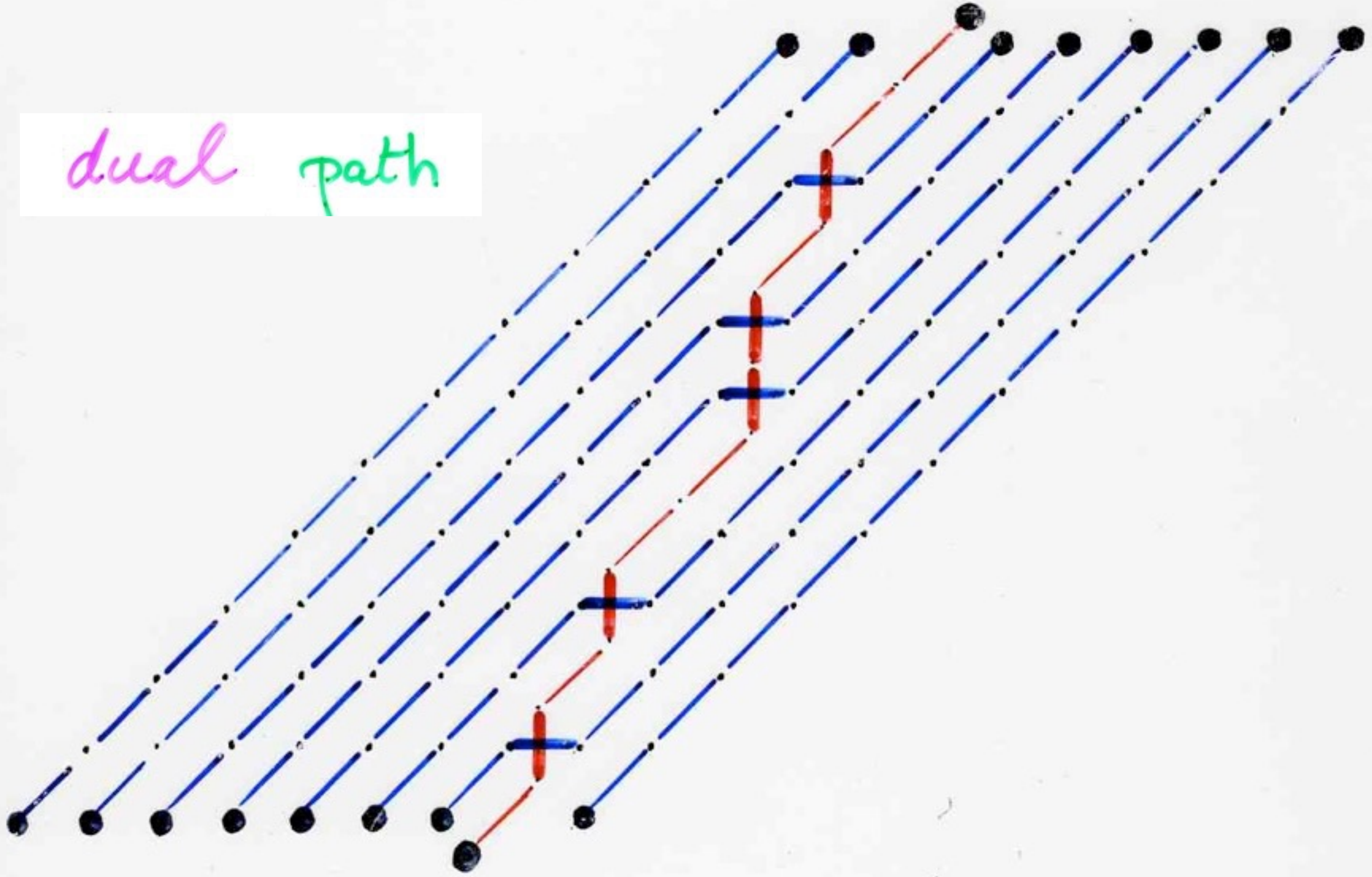
Part I, Ch 5b, 32-41



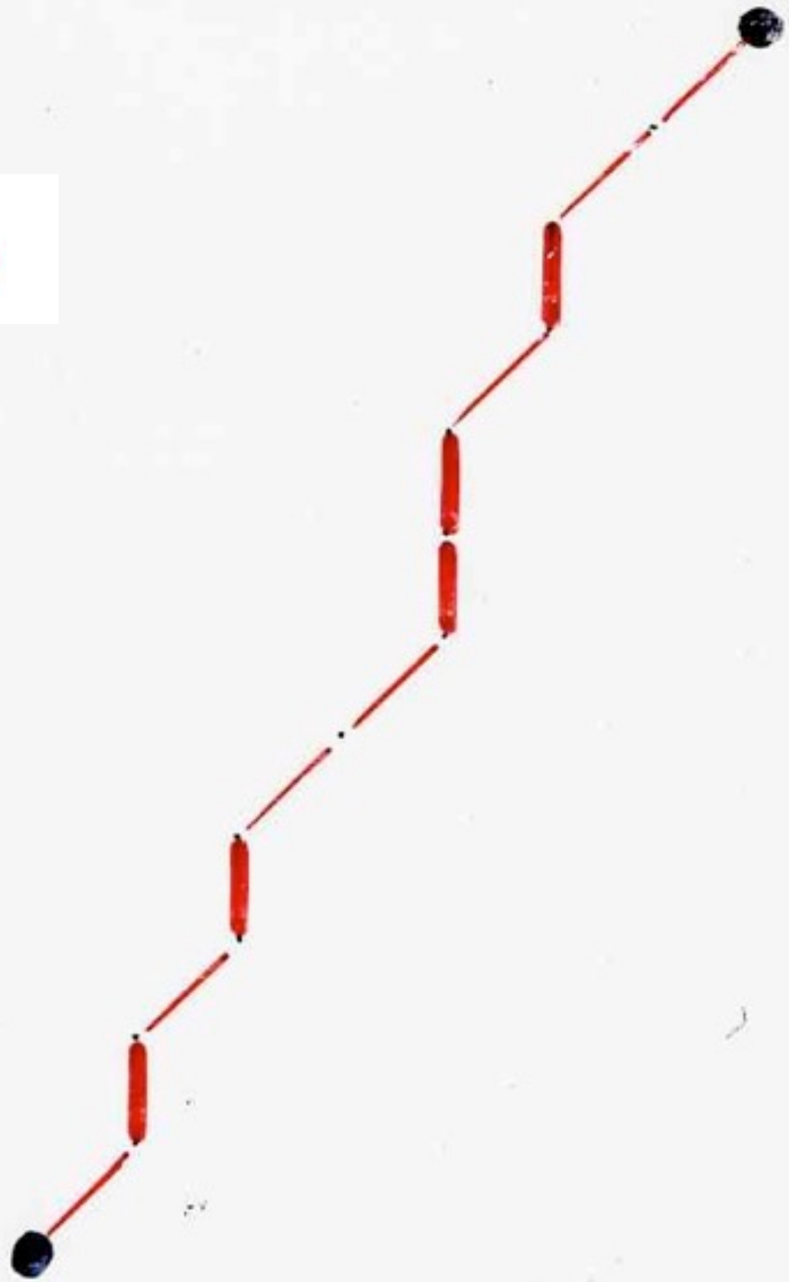



dual path

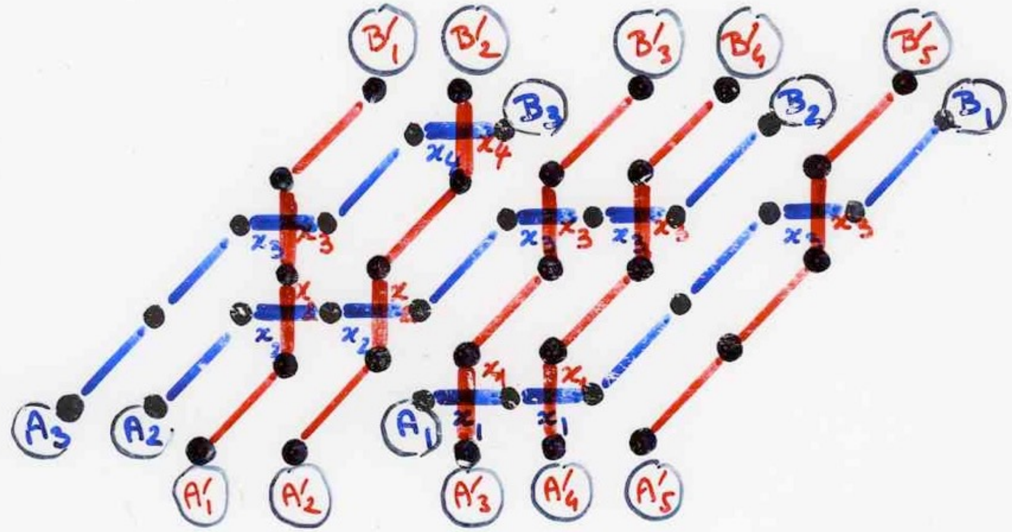


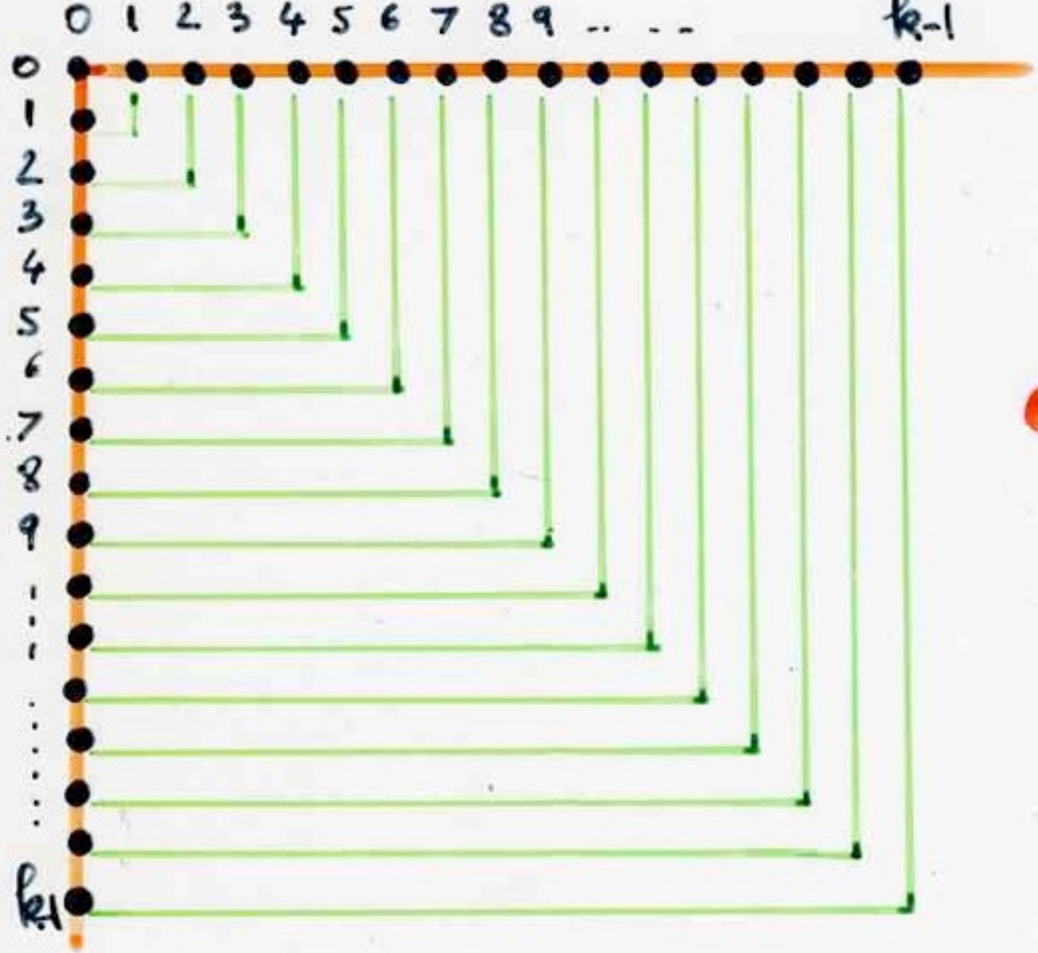

dual
path

dual path



dual configurations of non-intersecting paths





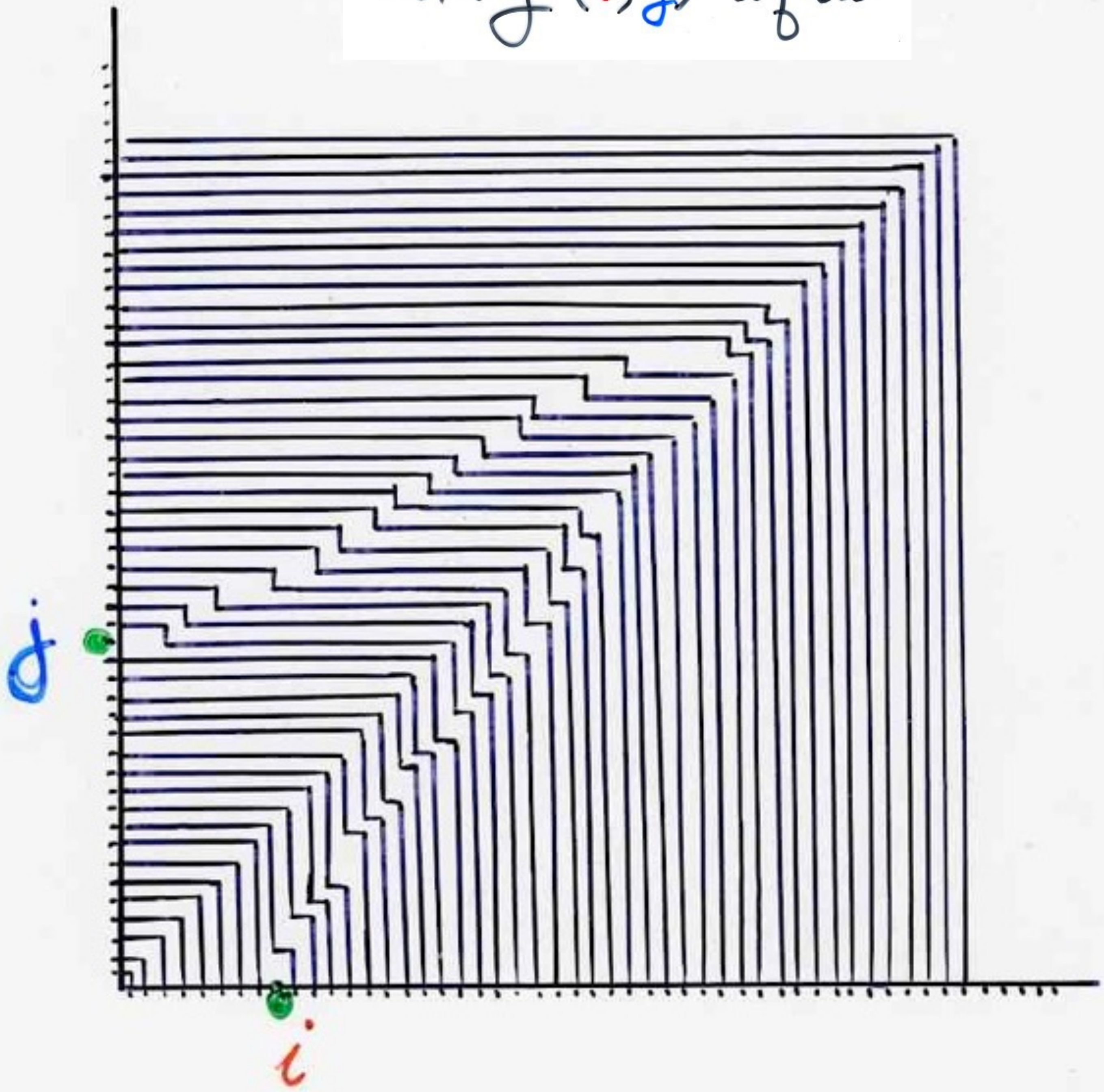
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & \binom{i+j}{i} & \\ 1 & 5 & & & & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

<u>exercise</u>

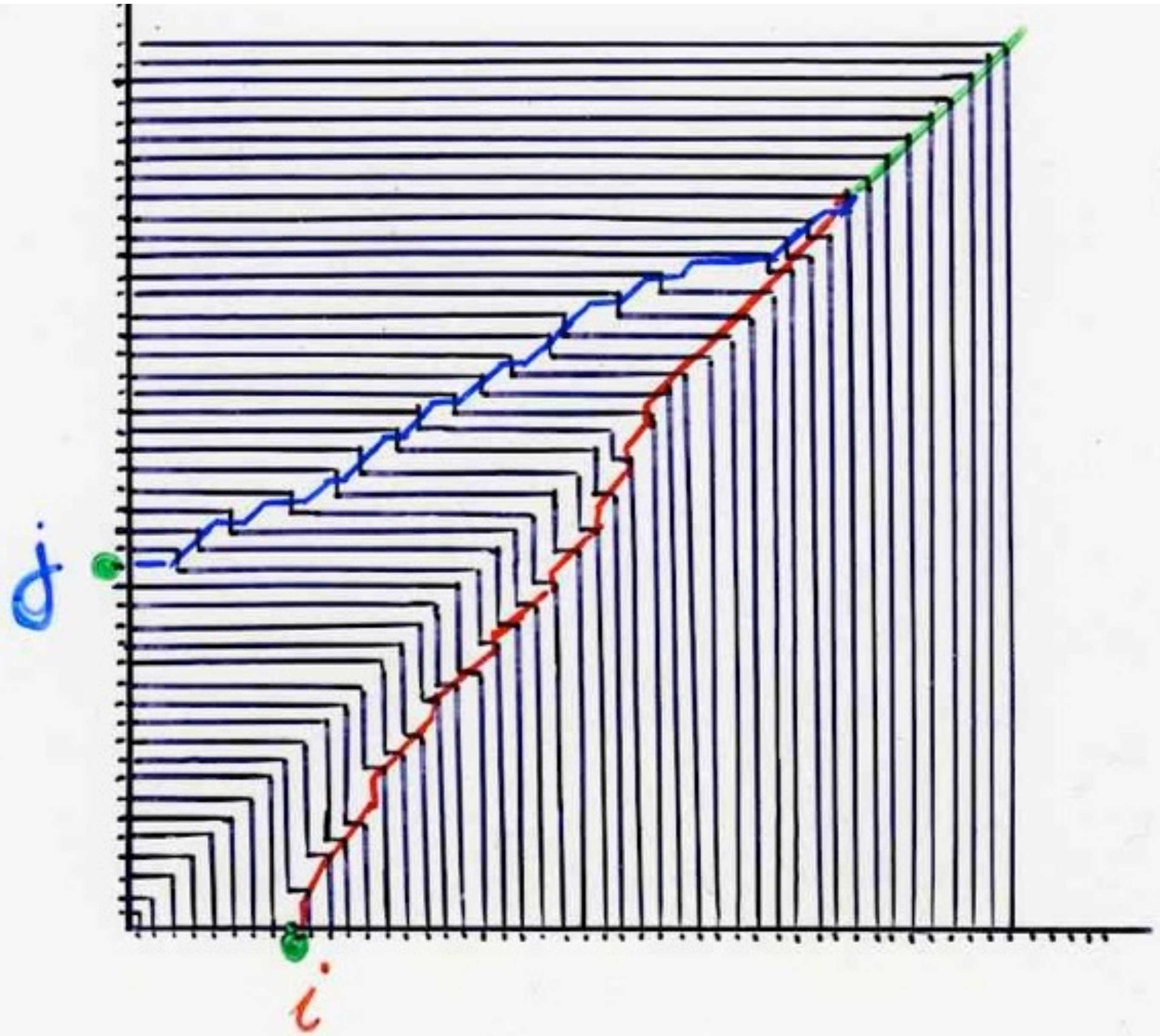
term (i, j) of the inverse matrix is

$$(-1)^{i+j} \sum_{k} \binom{k}{i}\binom{k}{j}$$

taking (i, j) cofactor



dual paths



$$(-1)^{i+j} \sum_{k} \binom{k}{i}\binom{k}{j}$$

giving a proof
of the formula
for the (i, j) cofactor

Complements

Inverse power series

Inversion in power series

$$f(t) = \sum_{n \geq 0} \mu_n t^n$$

$$f_{m,n} = \begin{vmatrix} \mu_m & \mu_{m-1} & \cdots & \mu_{m-n+1} \\ \mu_{m+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_{m-1} \\ \mu_{m+n-1} & \cdots & \mu_{m+1} & \mu_m \end{vmatrix}$$

$$= \det\left(\mu_{m+i-j}\right)_{1 \leq i,j \leq n}$$

$\mu_i = 0$ for $i < 0$

$$f_{m,n} = (-1)^{\frac{n(n-1)}{2}} \det\left(\mu_{m-n+1+i+j}\right)_{0 \leq i,j \leq n-1}$$

$$g(t) = \frac{1}{f(t)}$$

<u>Proposition</u> $$g_{n,m} = (-1)^{nm} f_{m,n}$$

for every $m,n \geq 1$

Idea of the proof

Suppose there exist $\{\lambda_k\}_{k\geq 1}$

$$\mu_n = \sum_{\substack{|\omega|=2n \\ \text{Dyck paths}}} v(\omega)$$

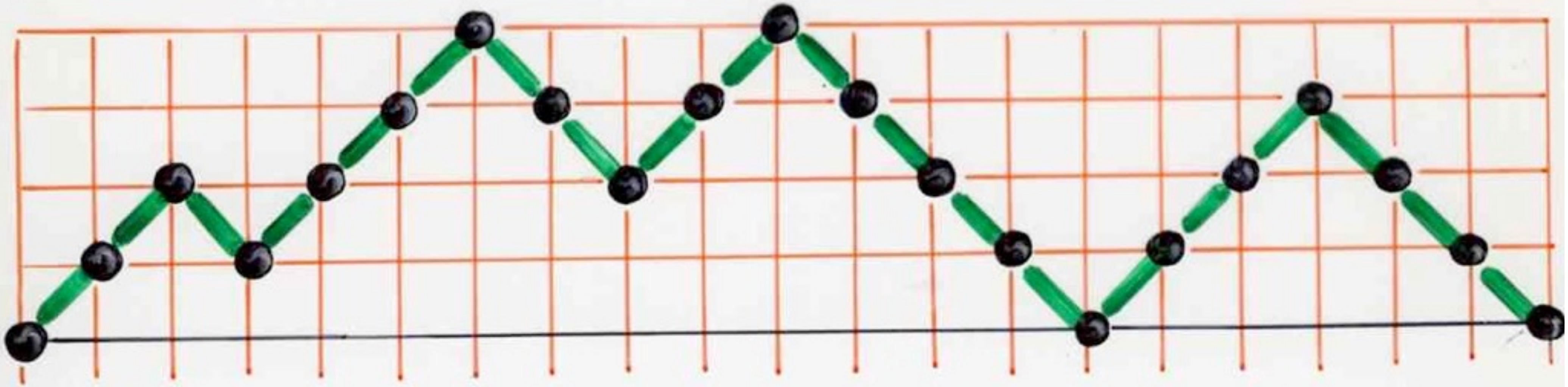
$$f(t) = f(t; \lambda_1, \ldots, \lambda_k, \ldots)$$

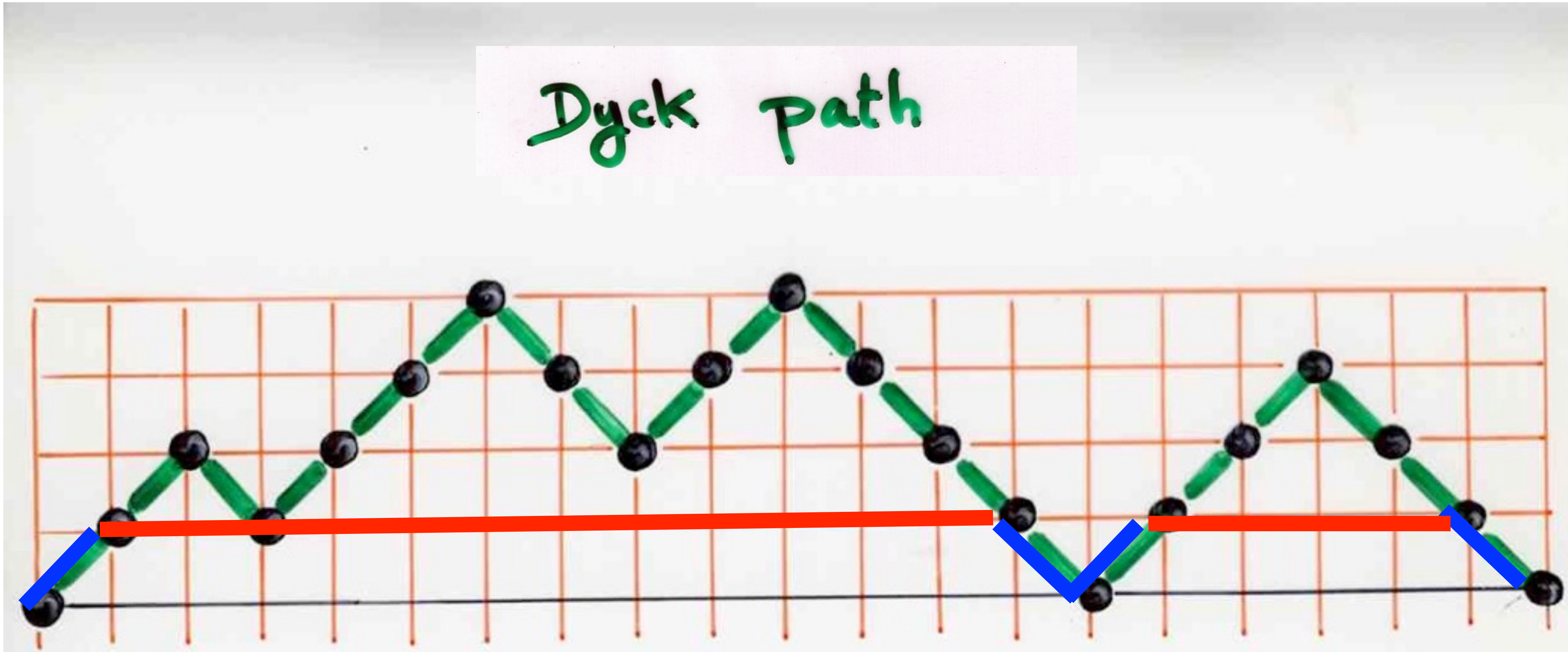
$$\delta f(t) = f(t; \lambda_2, \ldots, \lambda_{k+1}, \ldots)$$

$$\lambda_1 t \, \delta f(t) = \sum_{\substack{\omega \\ \text{prime Dyck path}}} v(\omega) \, t^{|\omega|/2}$$

$$f(t) = \frac{1}{1 - \lambda_1 t \, \delta f(t)}$$

Dyck path





Prime Dyck paths

(primitive)

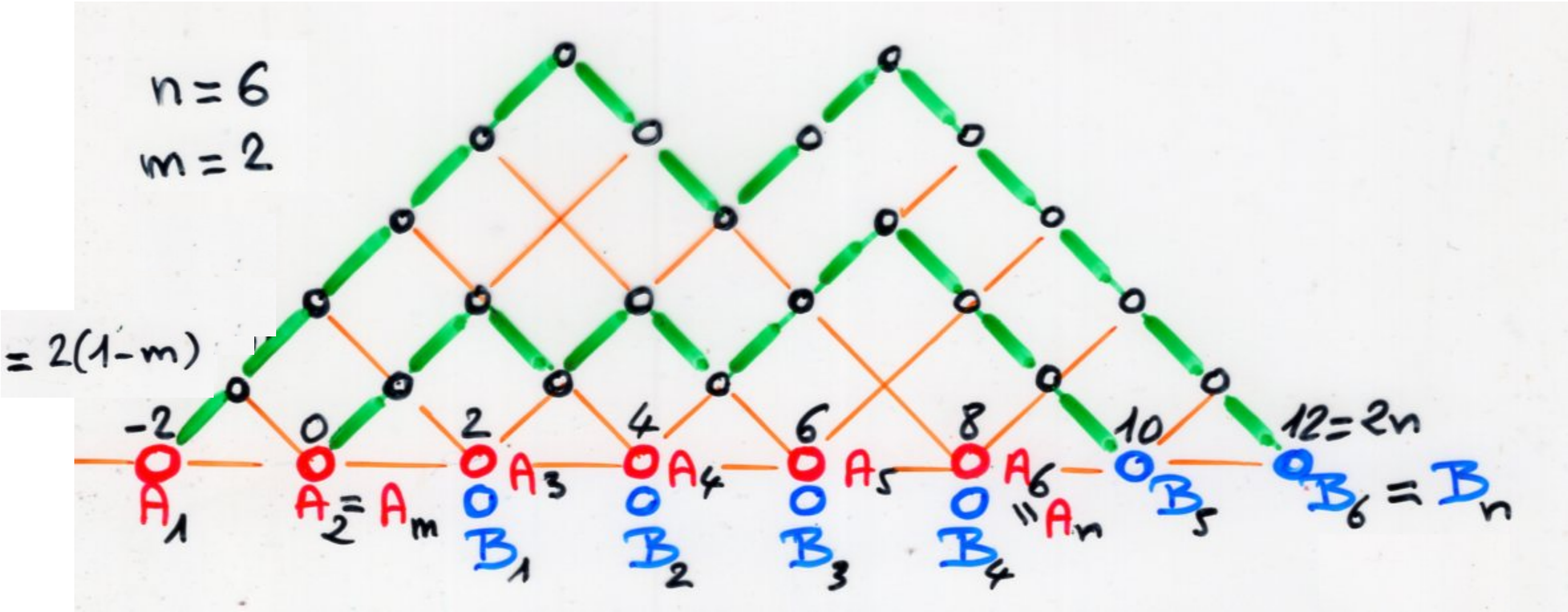
exercise

Prove the proposition

with configurations of non-crossing prime Dyck paths

solution: (in french) p IV 28 - 32

Lecture Notes X.V., Montreal, (1983)



The determinants $f_{m,n}$ and $g_{n,m}$

Complements

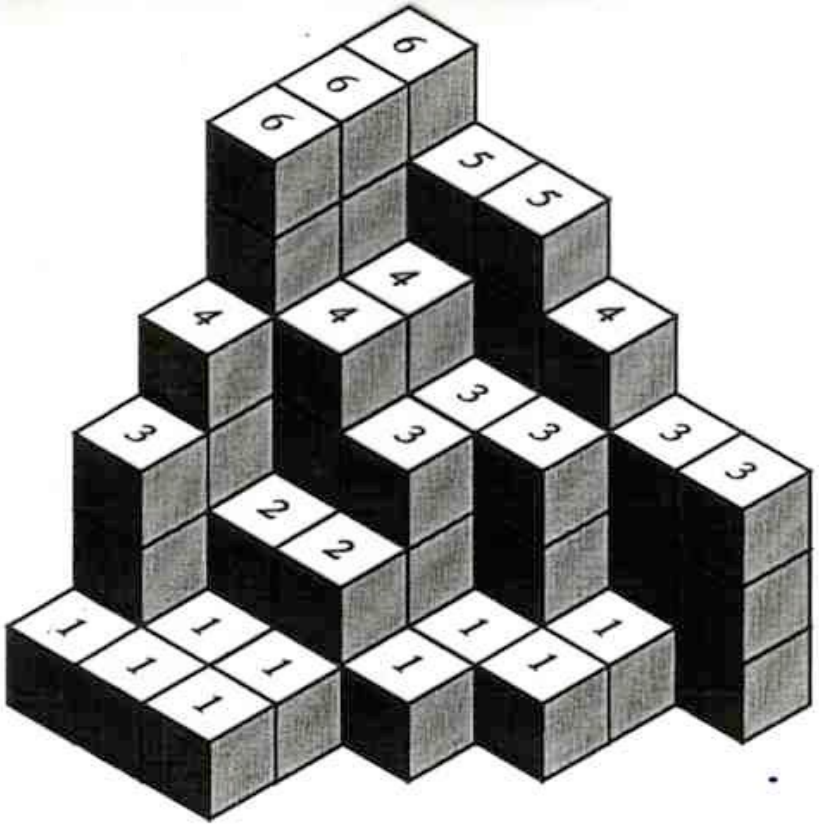
Some Hankel determinants

$$a_n = \frac{1}{3n+1}\binom{3n+1}{n}$$

- $$\Delta_n^{(0)} = \prod_{j=0}^{n-1} \frac{(3j+1)(6j)!\,(2j)!}{(4j+1)!\;(4j)!}$$

cyclically symm.
transpose-complement
plane partitions

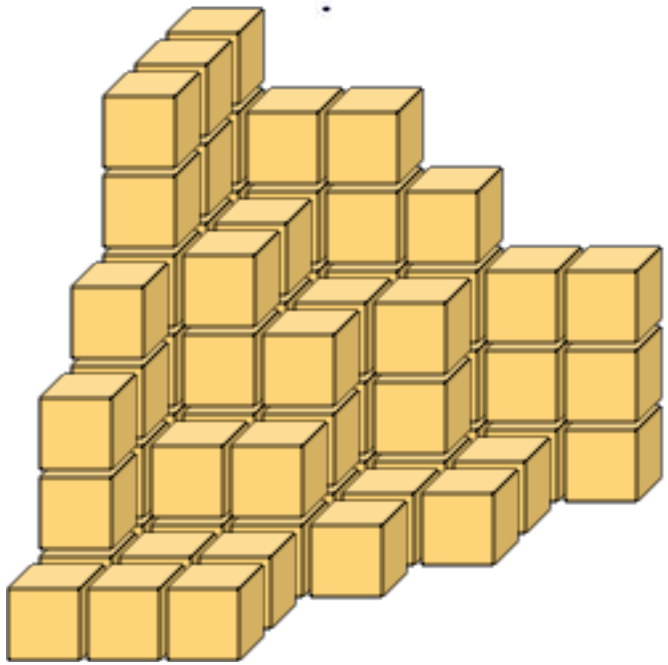
Tamm (2001)



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

plane partitions

cyclicaly symmetric plane partitions



$$a_n = \frac{1}{3n+1}\binom{3n+1}{n}$$

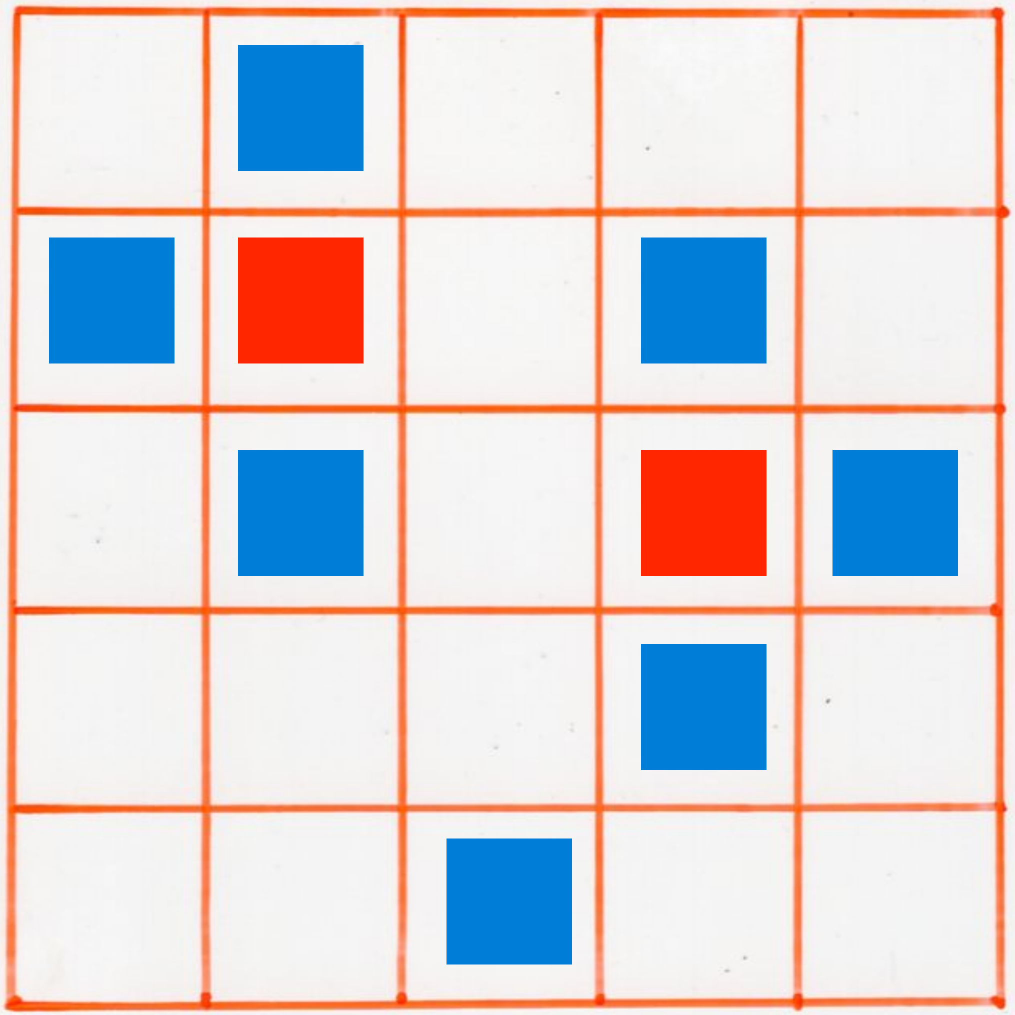
- $$\Delta_n^{(1)} = \prod_{j=1}^{n} \frac{\binom{6j-2}{2j}}{2\binom{4j-1}{2j}}$$

vertically symm.

alternating sign matrices

Tamm (2001)

alternating
sign
matrix



alternating
sign
matrix

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & \boxed{1} & 0 & 0 & 0 \\
\hline
\boxed{1} & \boxed{-1} & 0 & \boxed{1} & 0 \\
\hline
0 & \boxed{1} & 0 & \boxed{-1} & \boxed{1} \\
\hline
0 & 0 & 0 & \boxed{1} & 0 \\
\hline
0 & 0 & \boxed{1} & 0 & 0 \\
\hline
\end{array}$$

alternating
sign
matrix

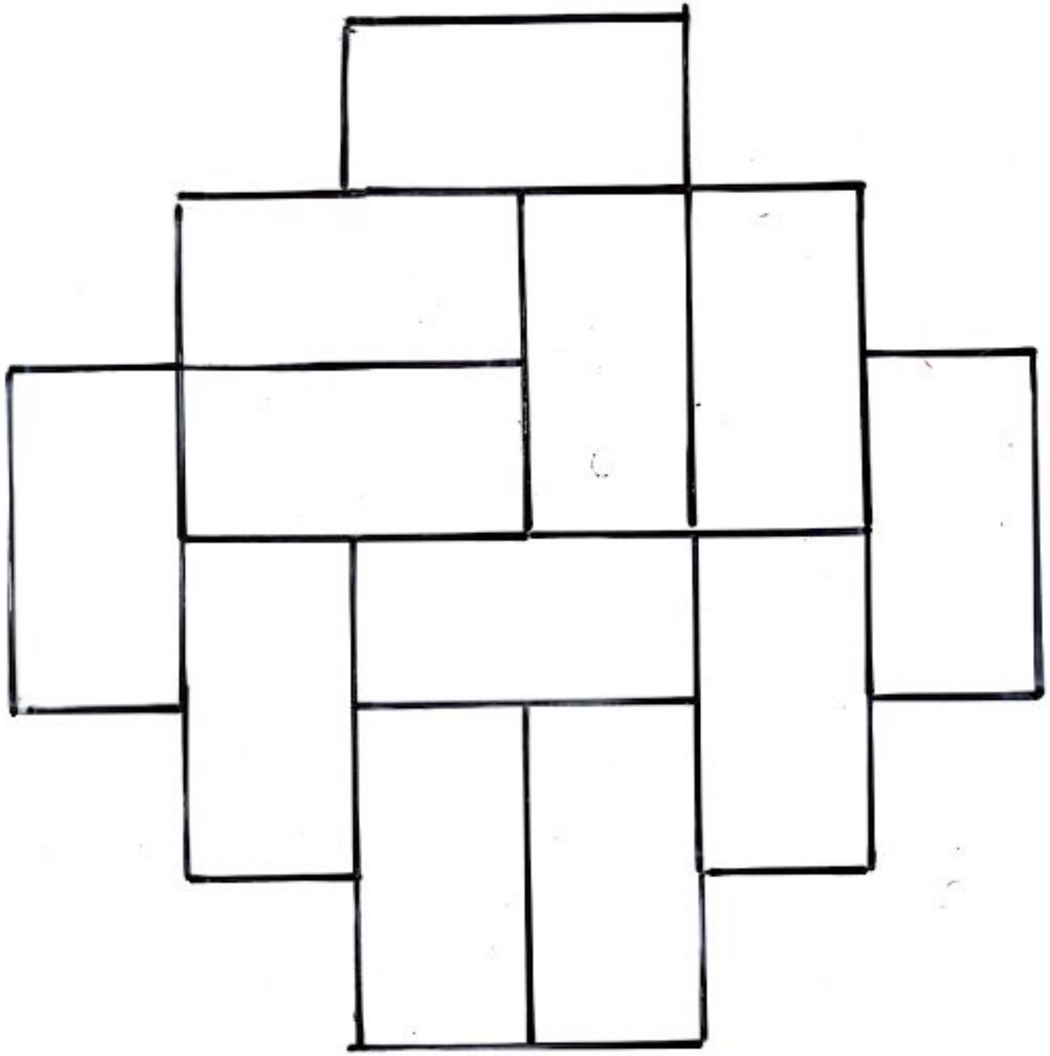
0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

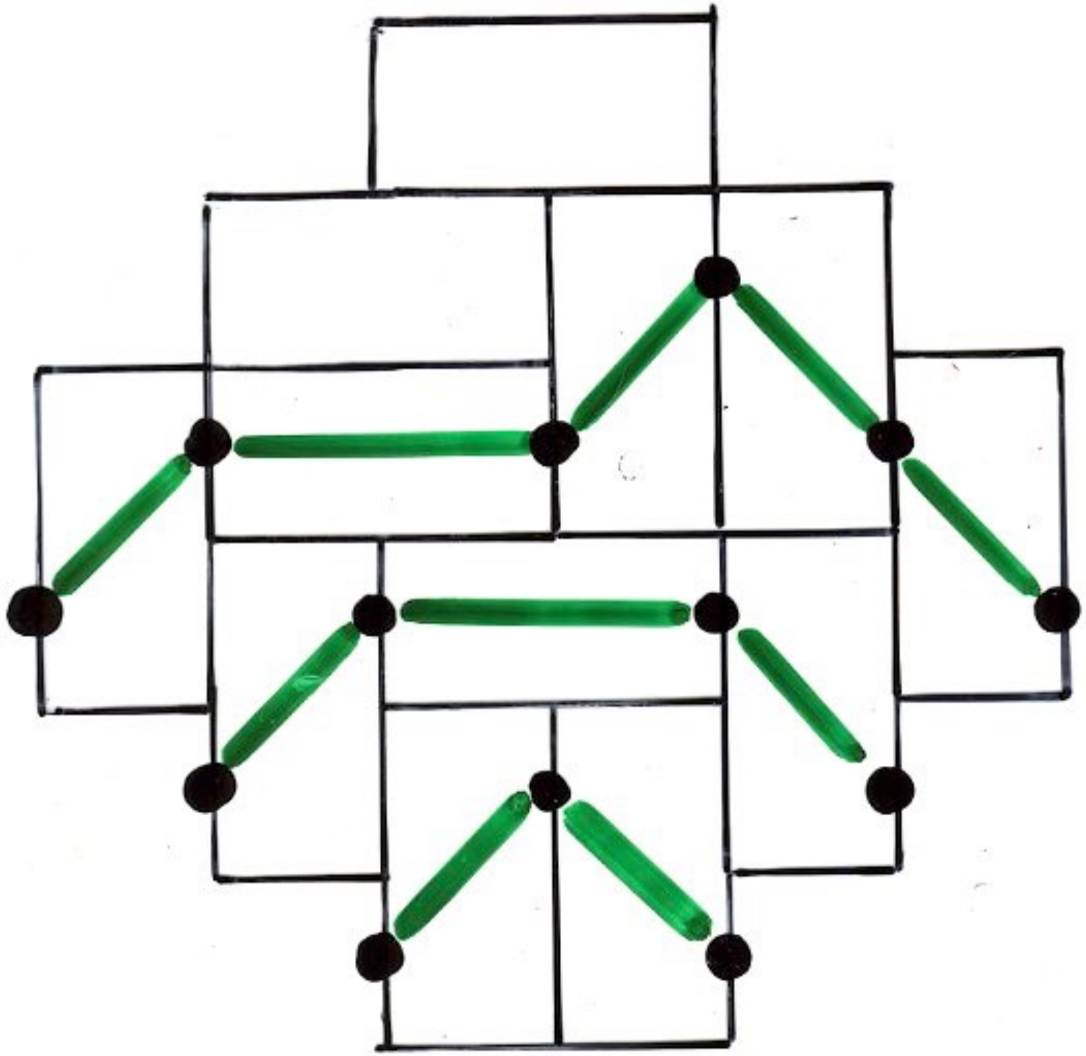
Hankel determinant

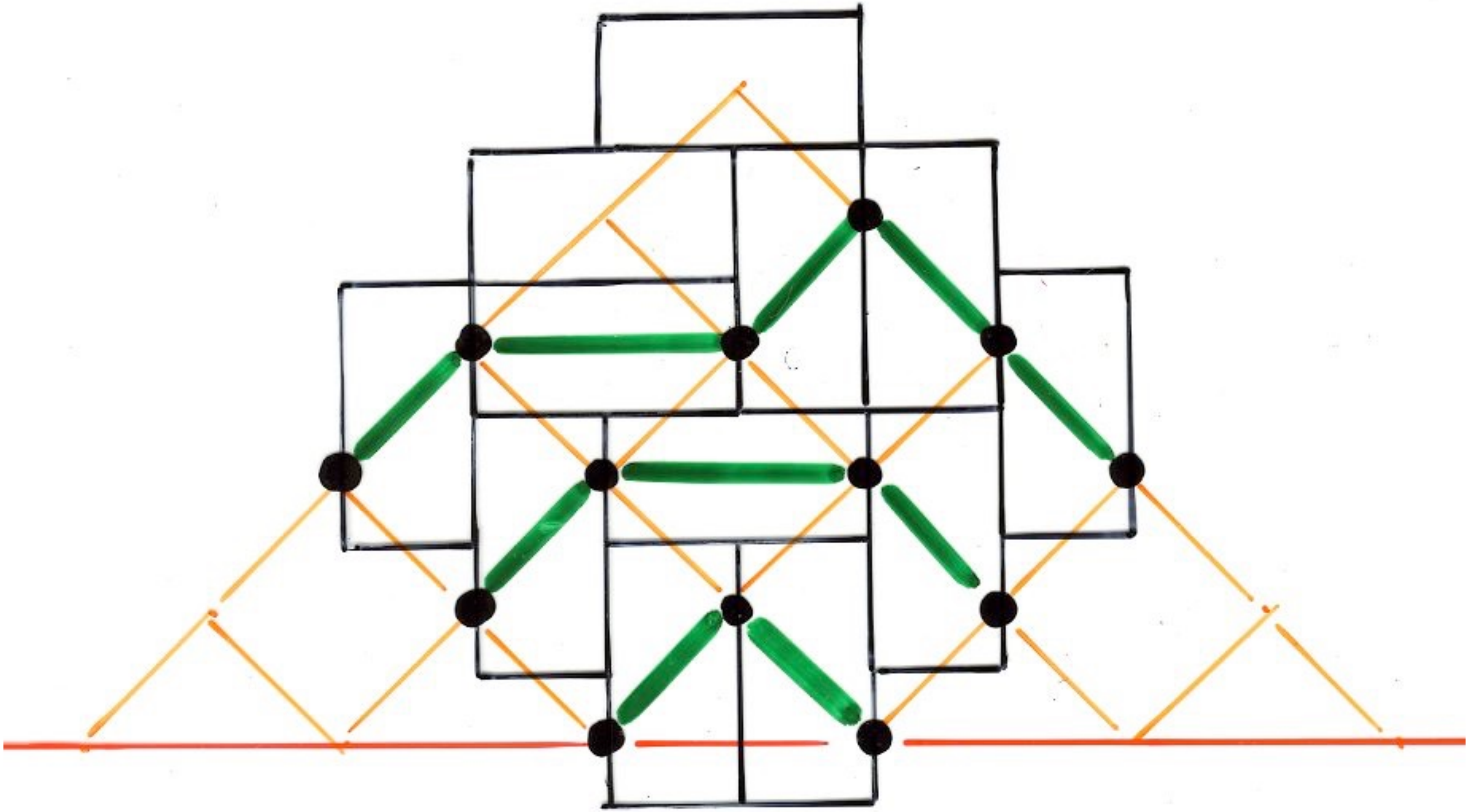
for

Aztec tilings

See Part I, Ch 5b, s87-113

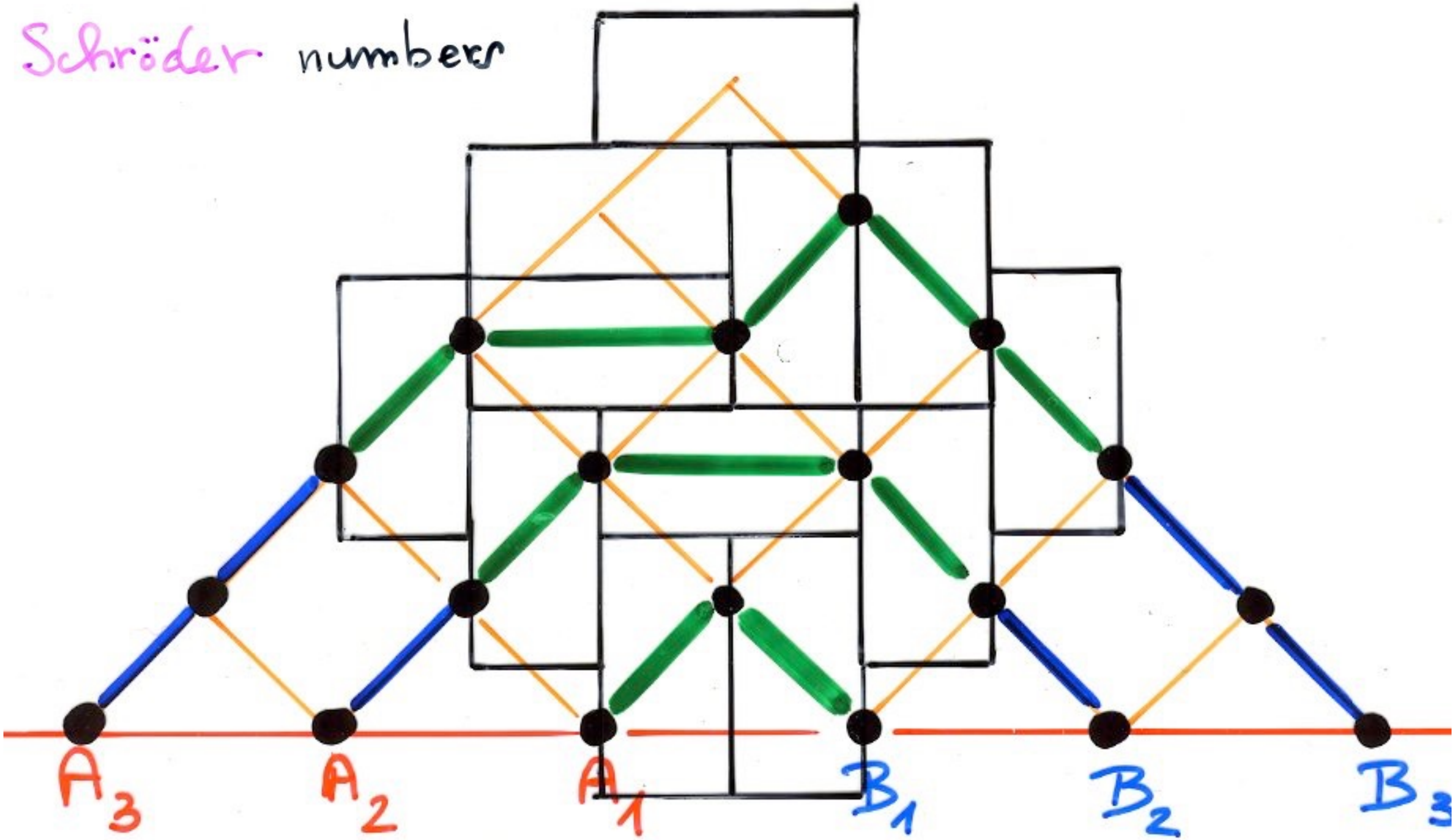






Schröder paths

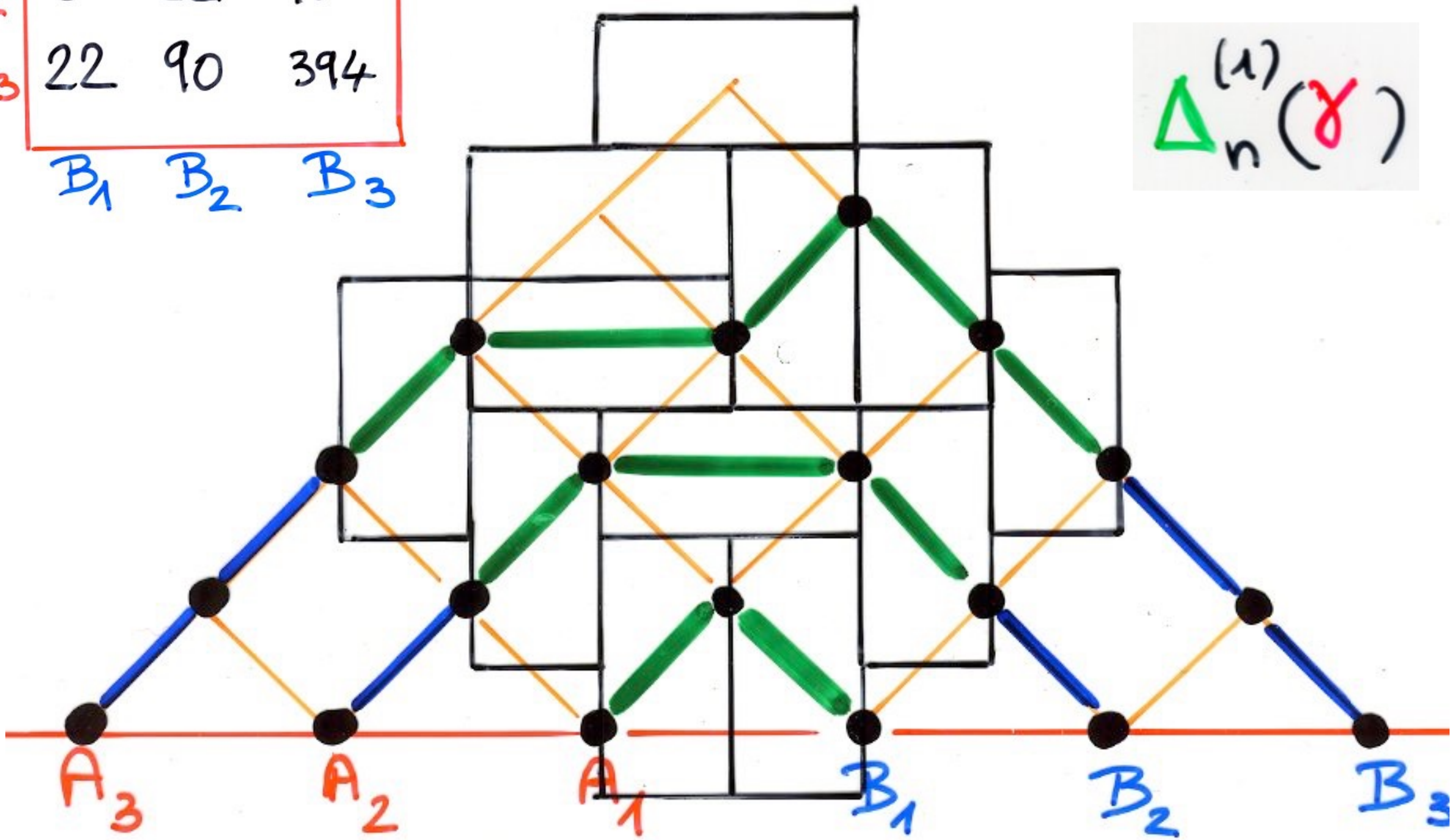
Schröder numbers



$$
\begin{array}{c|ccc|}
A_1 & 2 & 6 & 22 \\
A_2 & 6 & 22 & 90 \\
A_3 & 22 & 90 & 394 \\
\hline
 & B_1 & B_2 & B_3
\end{array}
$$

Hankel determinant

$$\Delta_n^{(1)}(\gamma)$$



$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2\times 22) - (6\times 6)$$
$$= 44 - 36$$

$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

$$= 44 - 36$$

$$= 8 = 2^3$$

(!)

$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$+\,17336 \qquad +\,11880 \qquad +\,11880 \;\rightarrow\; 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$-\,16200 \qquad -\,14184 \qquad -\,10648 \;\rightarrow\; -41032$$

$$\overline{64}$$

$$= 2^6$$

(!!)

« bijective computation »
of the Hankel determinant

of Schröder numbers giving
the number of tilings of the Aztec diagram

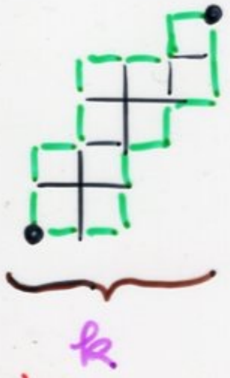
$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \beta^k$$

number of Dyck paths having k peaks

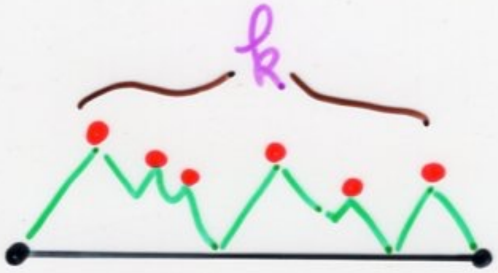
ω , $|\omega| = 2n$

$$\sum_{n \geq 0} \mu_{2n}(\beta) t^n = \cfrac{1}{1 - \cfrac{\beta t}{1 - \cfrac{t}{1 - \cfrac{\beta t}{1 - \cfrac{t}{\cdots}}}}}$$

$$\begin{cases} \lambda_{2k+1} = \beta \ , & k \geq 0 \\ \lambda_{2k} = 1 \ , & k \geq 1 \end{cases}$$

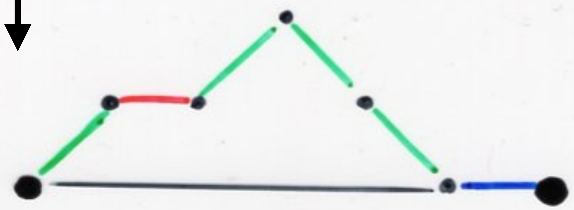


staircase polygons
number of columns



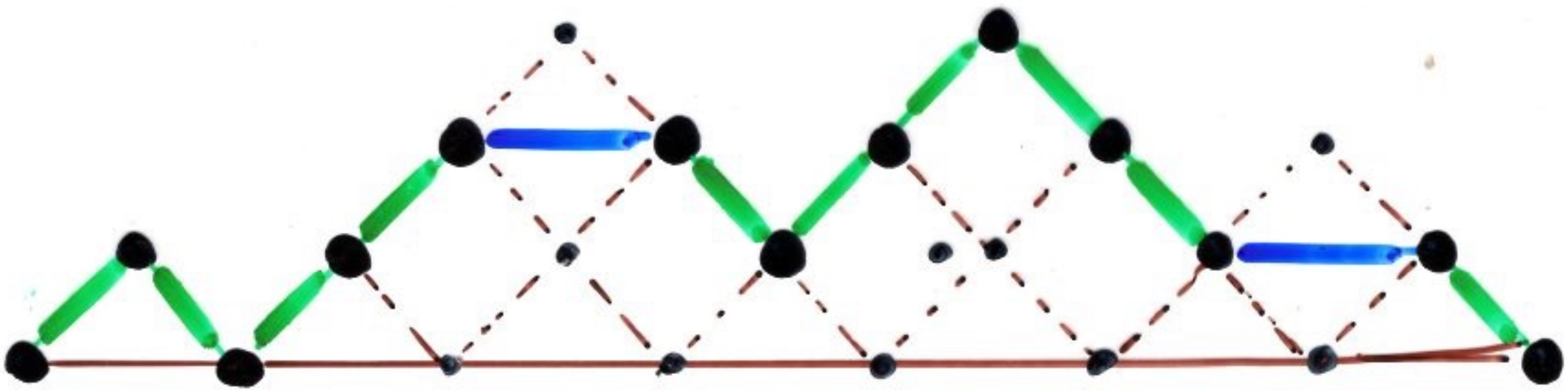
Dyck path
number of peaks

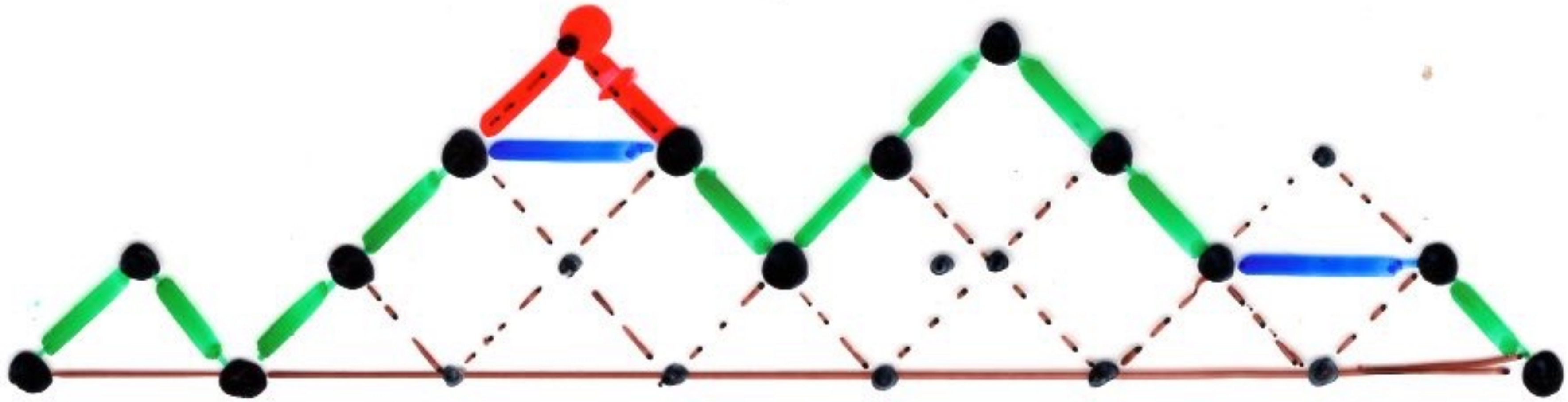
(β)- distribution $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$



number of / or —
in a 2-colored Motzkin path

Dyck path
number of SE steps
from odd to even level





(large)
Schröder
numbers

$$S(t) = \cfrac{1}{1 - 2t \cfrac{}{1 - t \cfrac{}{1 - 2t \cfrac{}{1 - t \cfrac{}{\cdots}}}}}$$

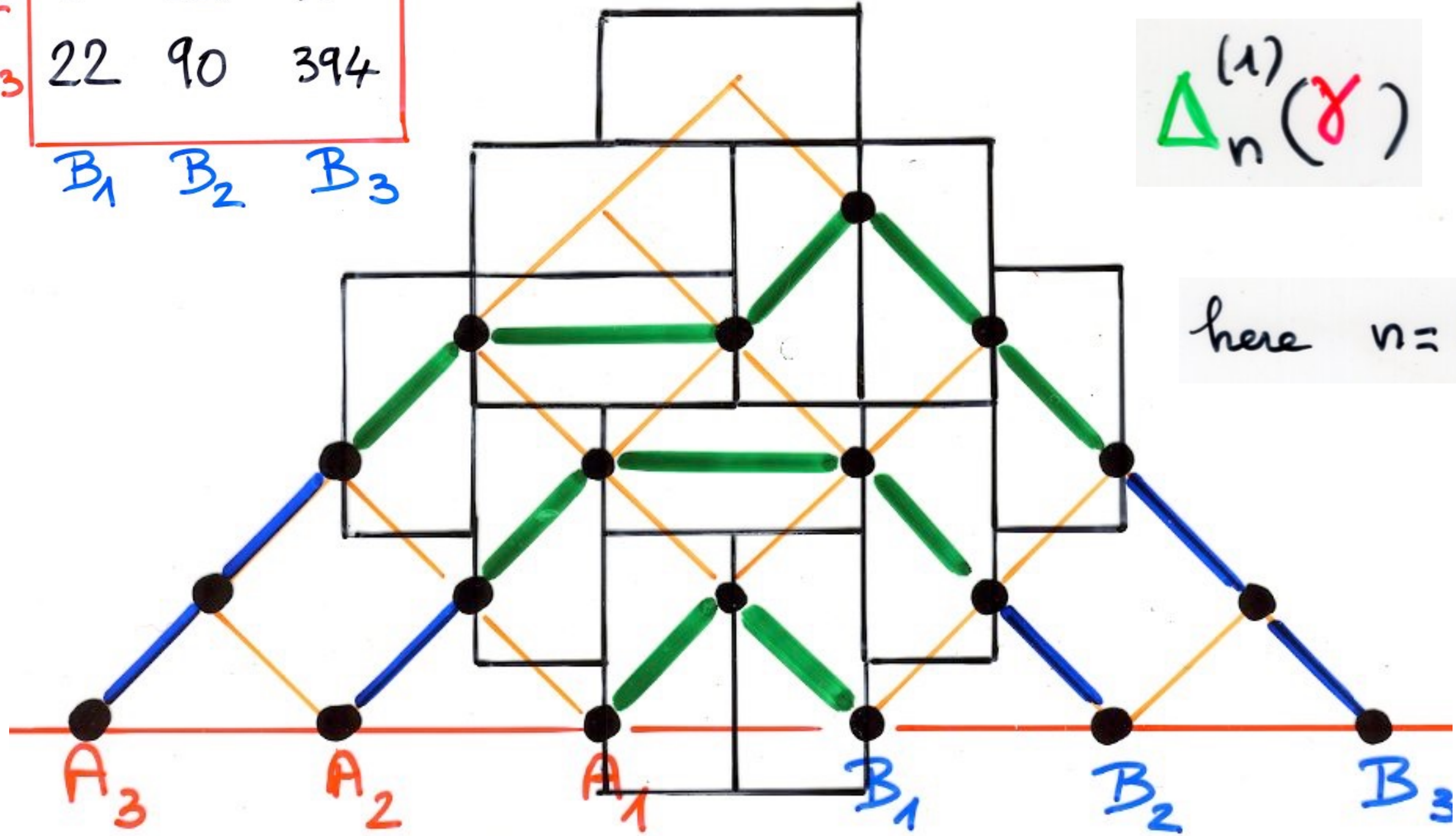
$$\begin{cases} \gamma_{2k+1} = 2, & k \geq 0 \\ \gamma_{2k} = 1, & k \geq 1 \end{cases}$$

$$\begin{array}{c|ccc|} A_1 & 2 & 6 & 22 \\ A_2 & 6 & 22 & 90 \\ A_3 & 22 & 90 & 394 \\ \hline & B_1 & B_2 & B_3 \end{array}$$

Hankel determinant

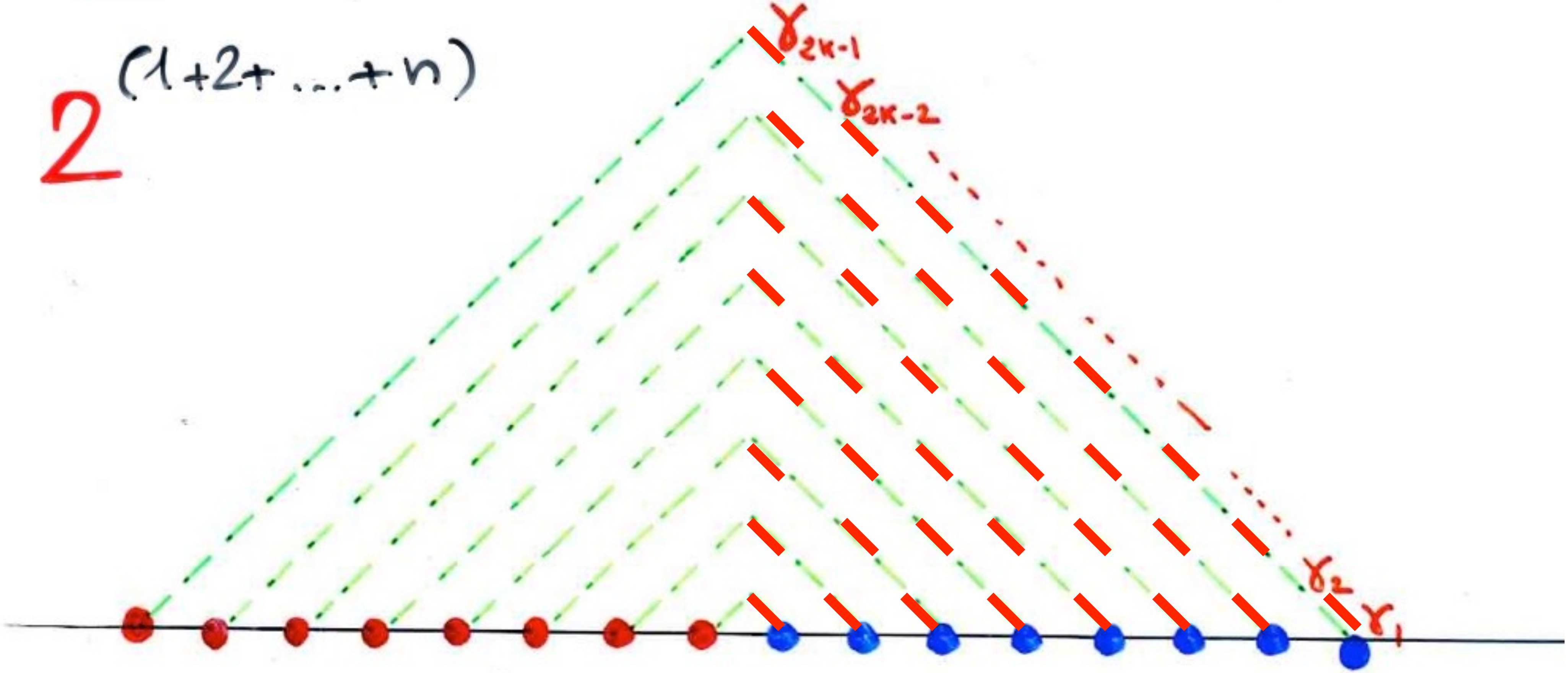
$$\Delta_n^{(1)}(\gamma)$$

here $n = 3$



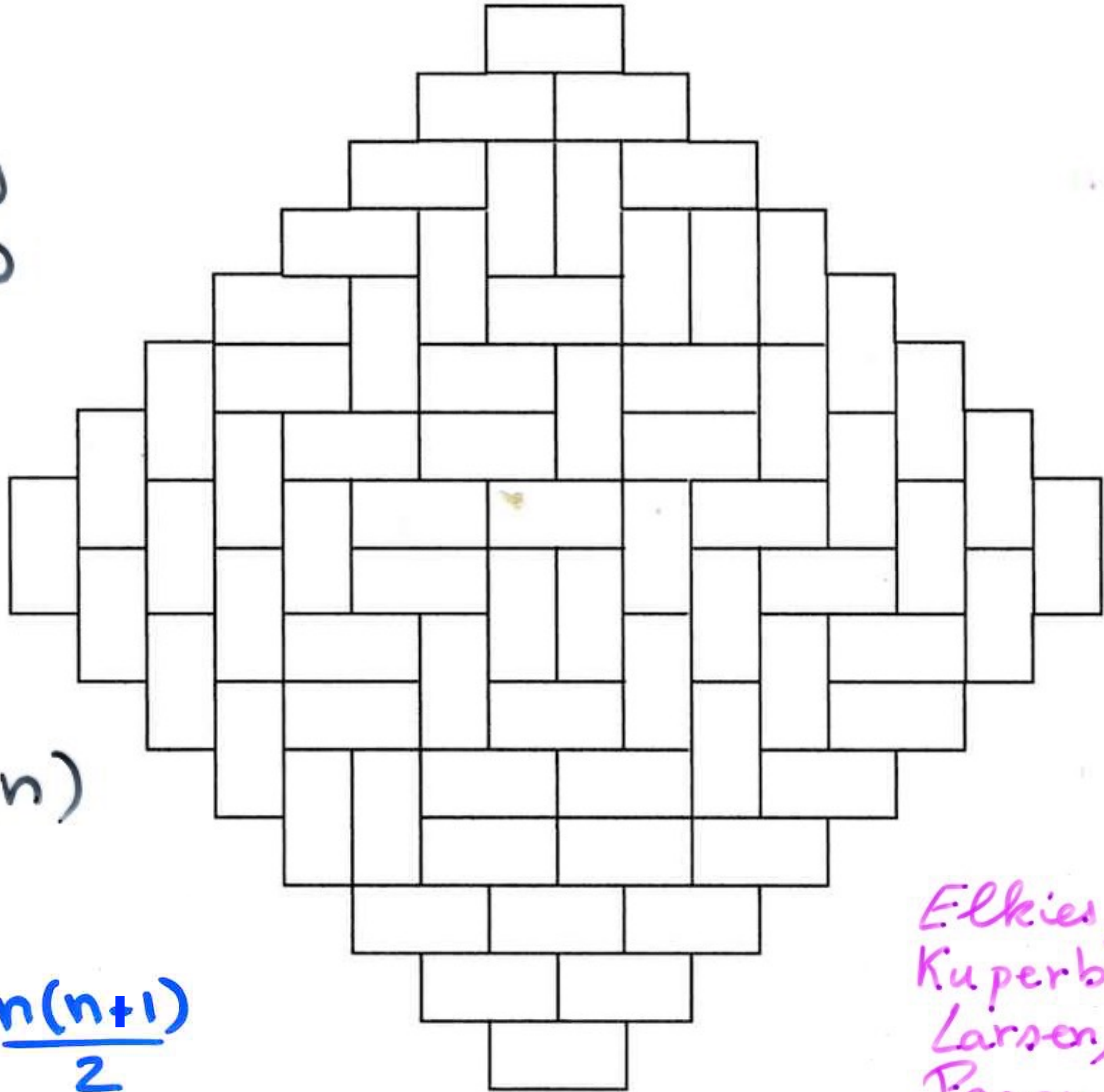
$$\Delta_n^{(1)}(\gamma) = H_v\begin{pmatrix} 1, \ldots, n \\ 1, \ldots, n \end{pmatrix}$$

$$2^{(1+2+\ldots+n)}$$



number of

tilings

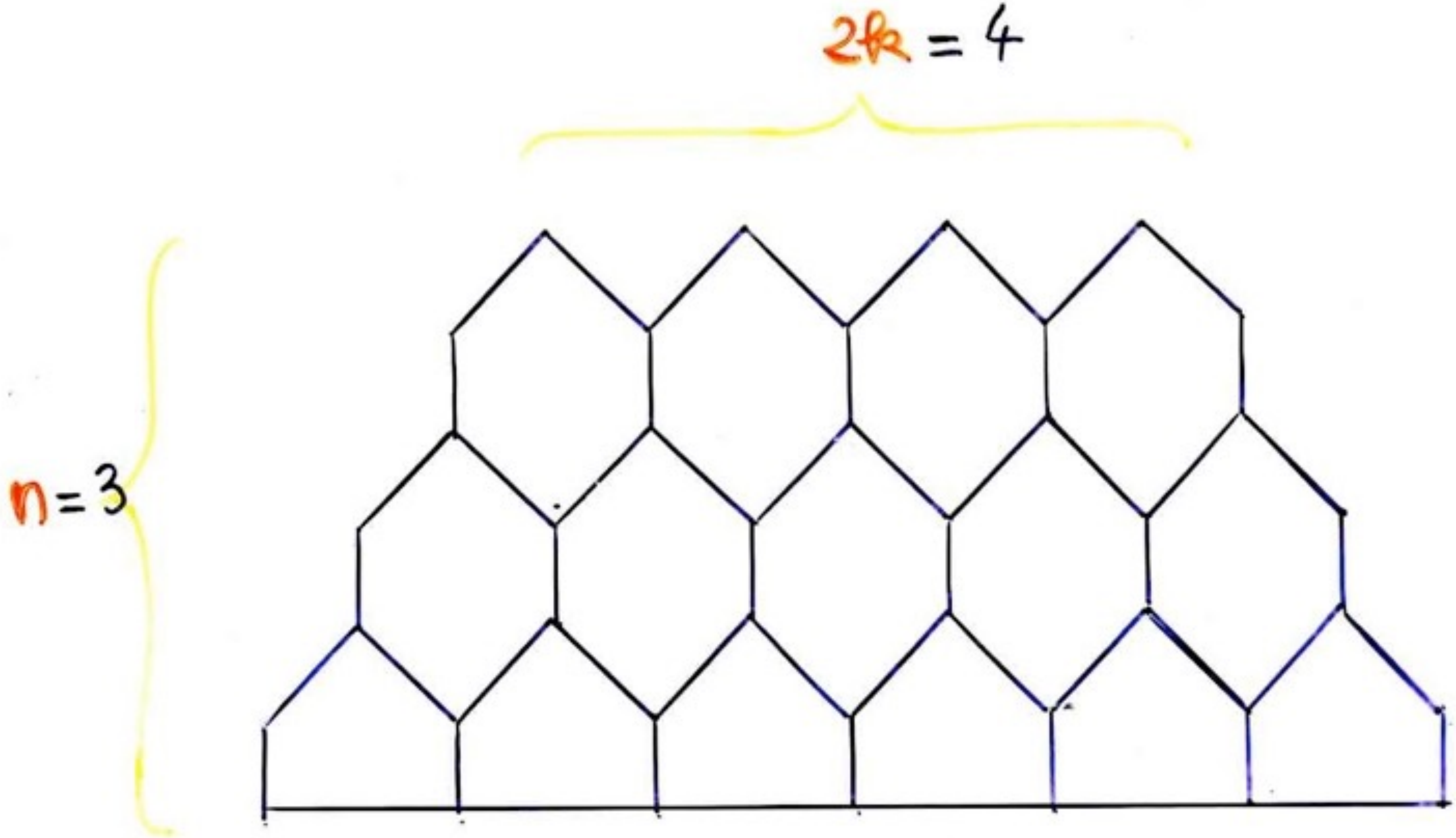


$$2^{(1+2+\ldots+n)}$$

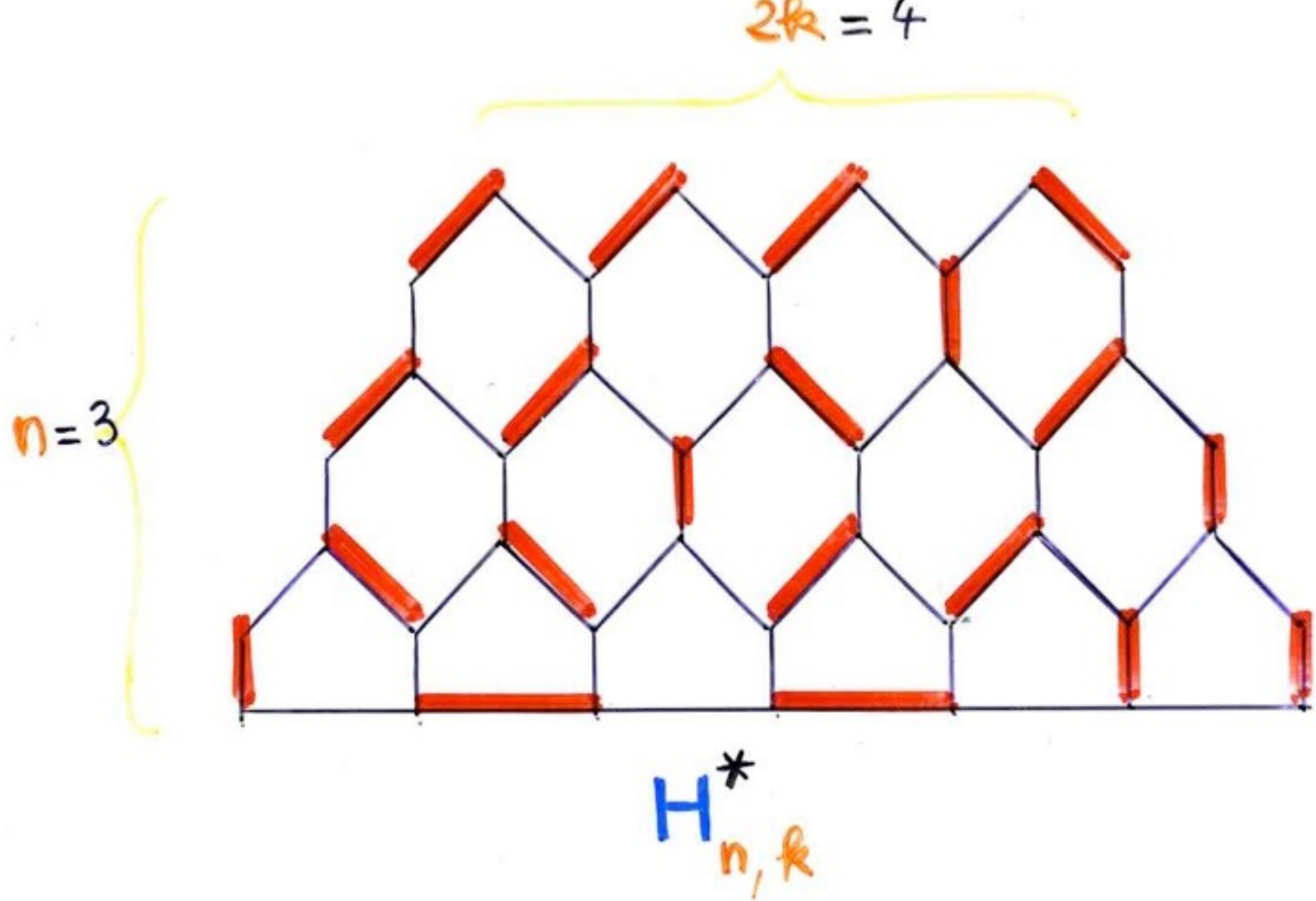
$$2^{\frac{n(n+1)}{2}}$$

Elkies,
Kuperberg,
Larsen,
Propp
(1992)

Another Hankel determinant



$H^{*}_{n,k}$



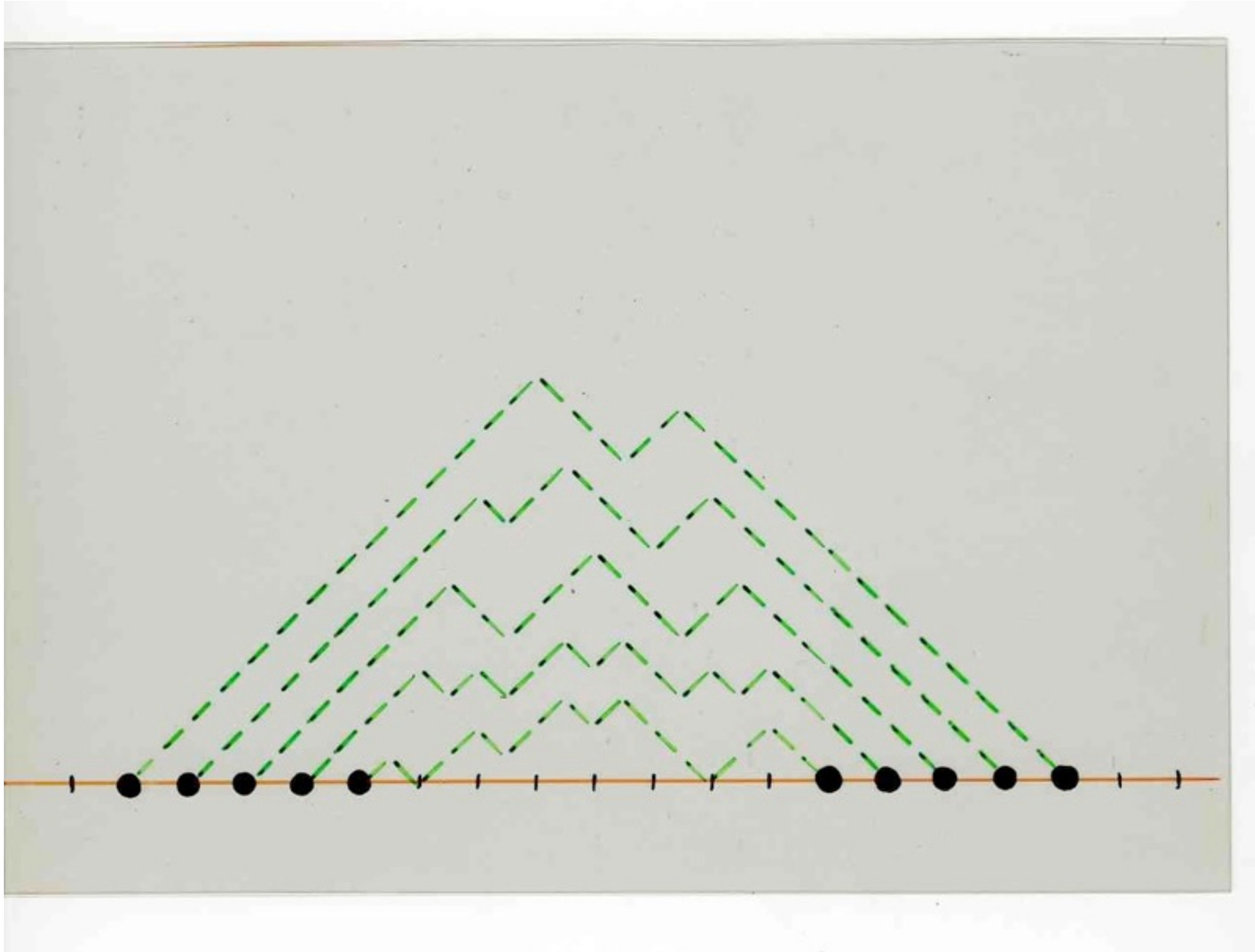
$2k = 4$
$n = 3$
$H^*_{n,k}$

number of perfect matchings of $H^{*}_{n,k}$ $= \displaystyle\prod_{1\leq i\leq j\leq n} \frac{(i+j+2k)}{(i+j)}$

de Sainte-Catherine, X.V. (1985)

$$\begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+k-1} \\ C_{n+1} & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ C_{n+k-1} & \cdots & \cdots & C_{n+2k-2} \end{vmatrix} = \prod_{1 \leq i < j \leq n} \frac{(i+j+2k)}{(i+j)}$$

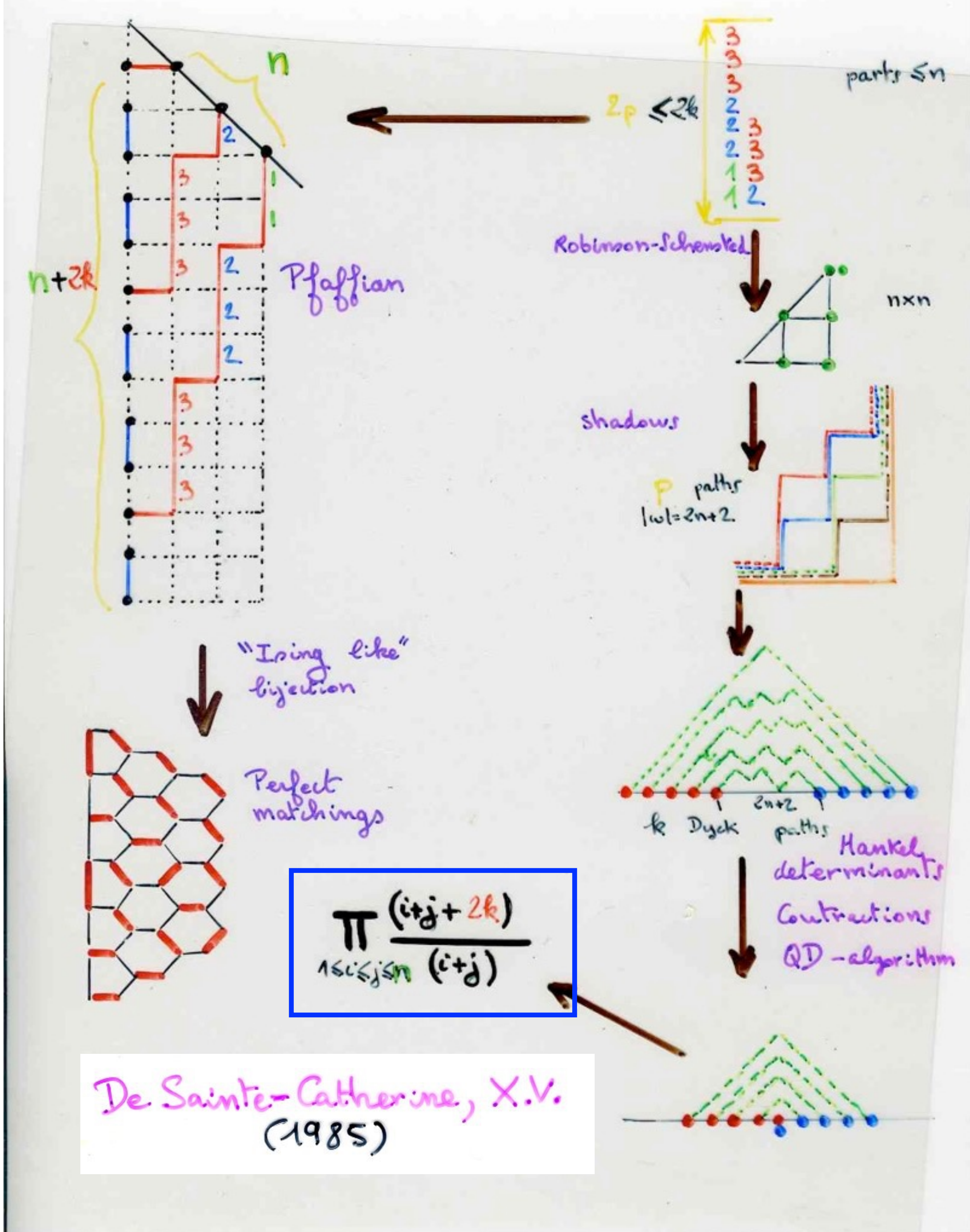
Hankel
determinant
of
Catalan
numbers



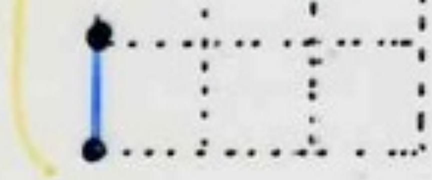
a nice formula ....

with a festival of bijections

Part I, Ch 5b, epilogue


n
n+2k
Pfaffian
2p ≤ 2k
parts ≤ n
Robinson-Schensted
n×n
shadows
p paths
|ω| = 2n+2
"Ising like" bijection
Perfect matchings
k Dyck paths
2n+2
Hankel determinants
Contractions
QD-algorithm
$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$
De Sainte-Catherine, X.V.
(1985)

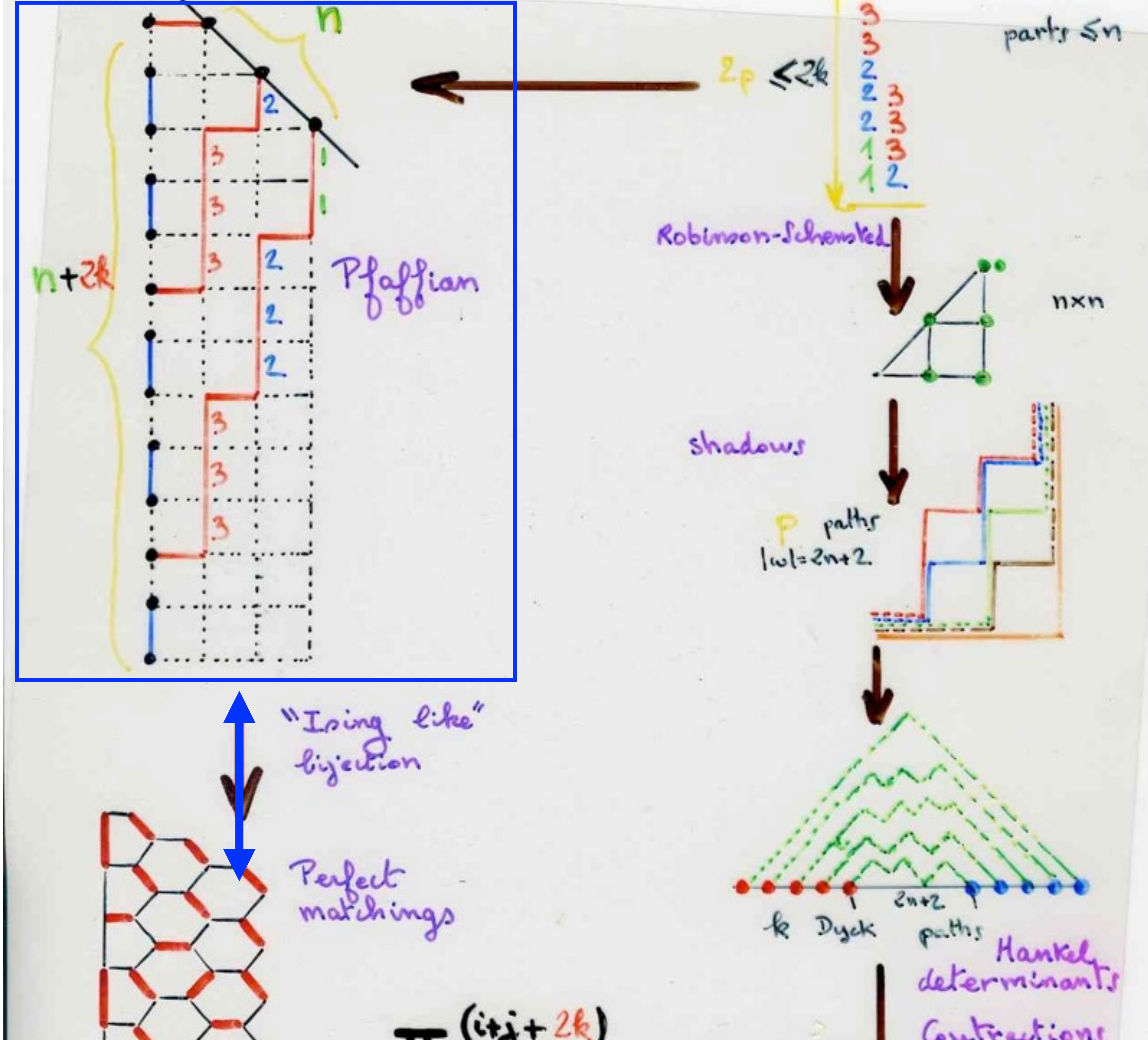
2022



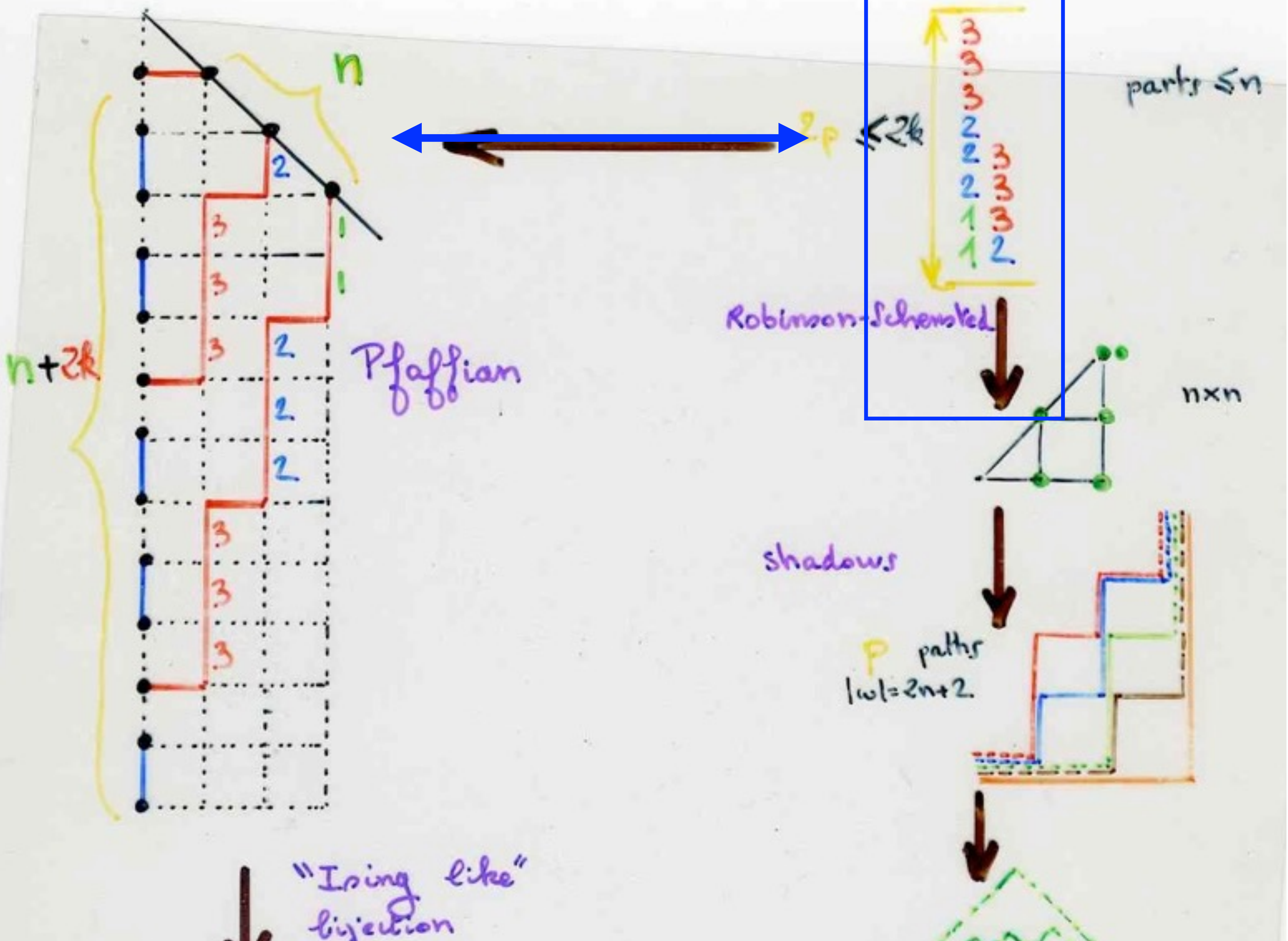
"Ising like" bijection

Perfect matchings

$$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$$



n
n+2k
Pfaffian
$2p \leq 2k$
parts $\leq n$
Robinson-Schensted
n×n
shadows
p paths
$|w| = 2n+2$
k Dyck paths
2n+2
Hankel determinants
"Ising like" bijection
Perfect matchings


n
n+2k
Pfaffian
"Ising like"
bijection
2p ≤2k
parts ≤n
Robinson-Schensted
n×n
shadows
p paths
|ω|=2n+2

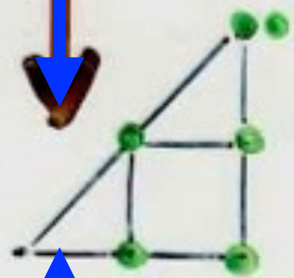
parts $\leq n$

$2p \leq 2k$

3
3
3
2
2 3
2 3
1 3
1

Robinson-Schensted

...ffian



$n \times n$

shadows

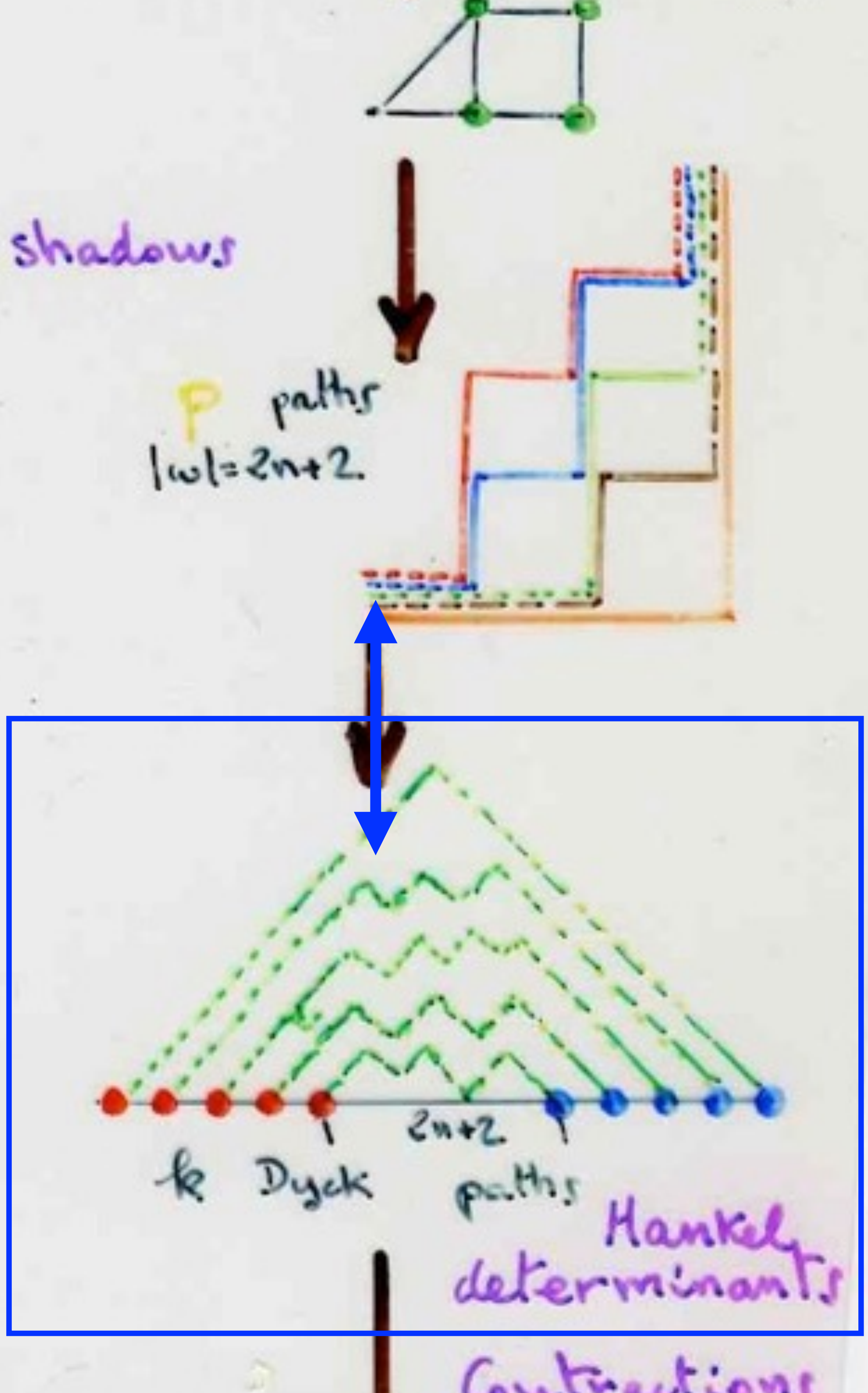
p paths

$|\omega| = 2n+2$

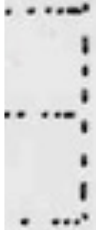
ℓike"

ngs

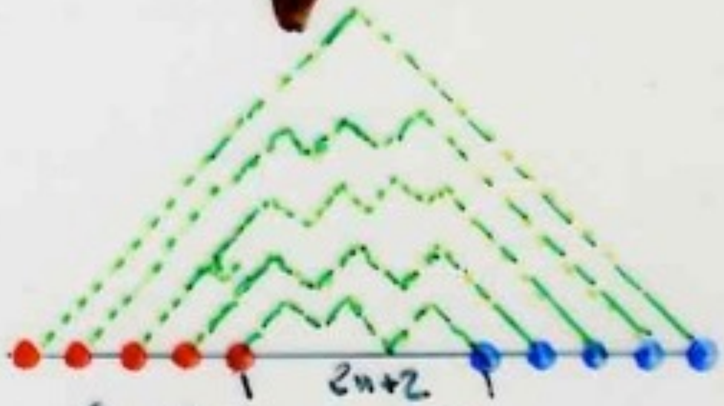
$(i+j+2k)$



$|\omega| = 2n+2$



"Ising like" bijection



Perfect matchings

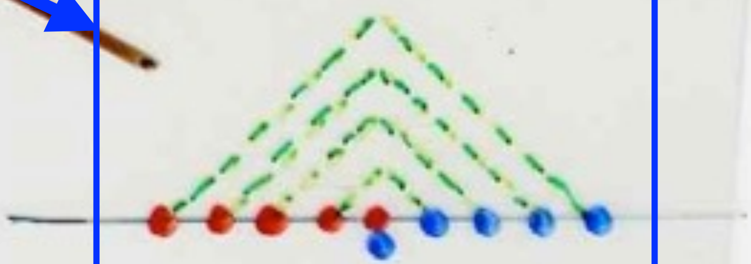
k Dyck paths

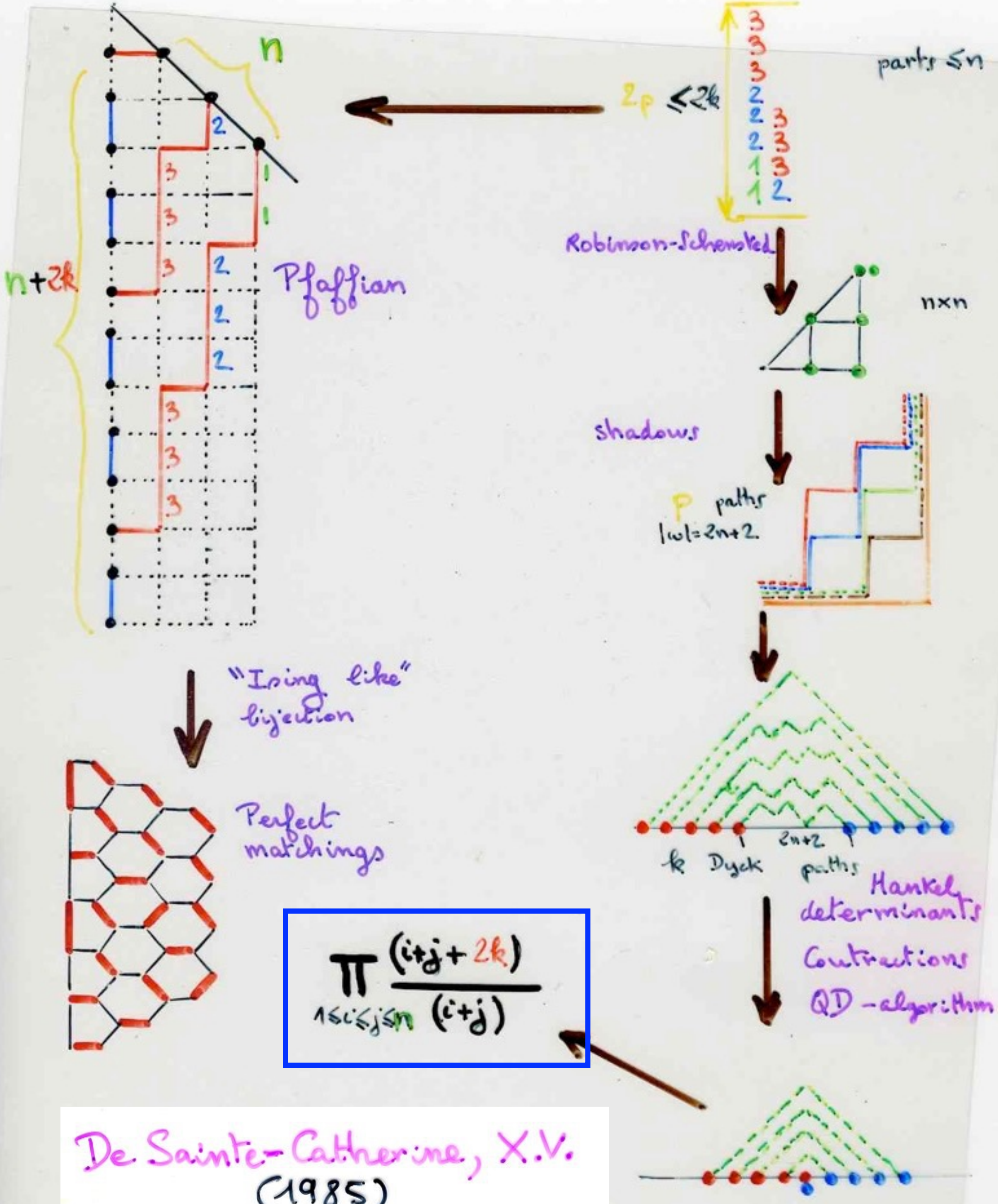
Hankel determinants

Contractions

QD-algorithm

$$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$$





De Sainte-Catherine, X.V.
(1985)

$$\begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+k-1} \\ C_{n+1} & \cdots & & \vdots \\ \vdots & & & \vdots \\ C_{n+k-1} & \cdots & & C_{n+2k-2} \end{vmatrix} = \prod_{1\leq i\leq j\leq n} \frac{(i+j+2k)}{(i+j)}$$

Hankel determinant of Catalan numbers

q-d algorithm

quotient-difference algorithm

See next chapter: Ch 4b