An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 0
Introduction to the course

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enumerative combinatorics

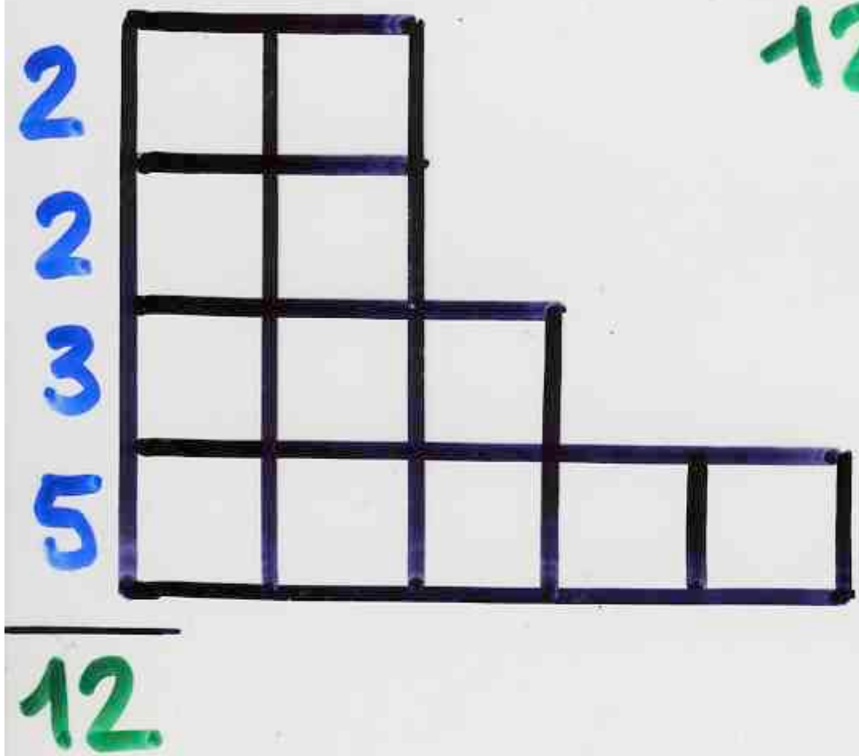
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

number of permutations on $\{1, 2, \ldots, n\}$ $= 1 \times 2 \times 3 \times \ldots \times n$

$= n!$

an example with Young tableaux

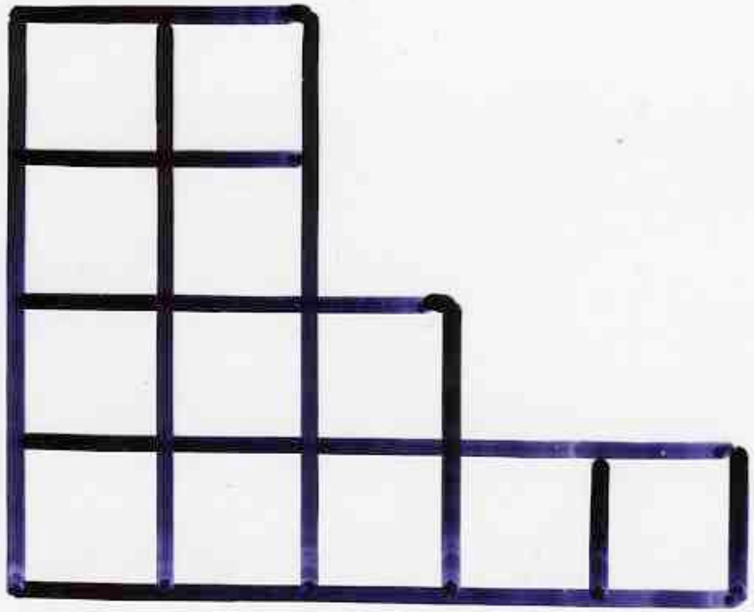
$12 = n = 5 + 3 + 2 + 2$



Ferrers diagram

Partition of n

λ



7	12			
6	10			
3	5	9		
1	2	4	8	11

Young tableau

shape λ

f_λ = nb of

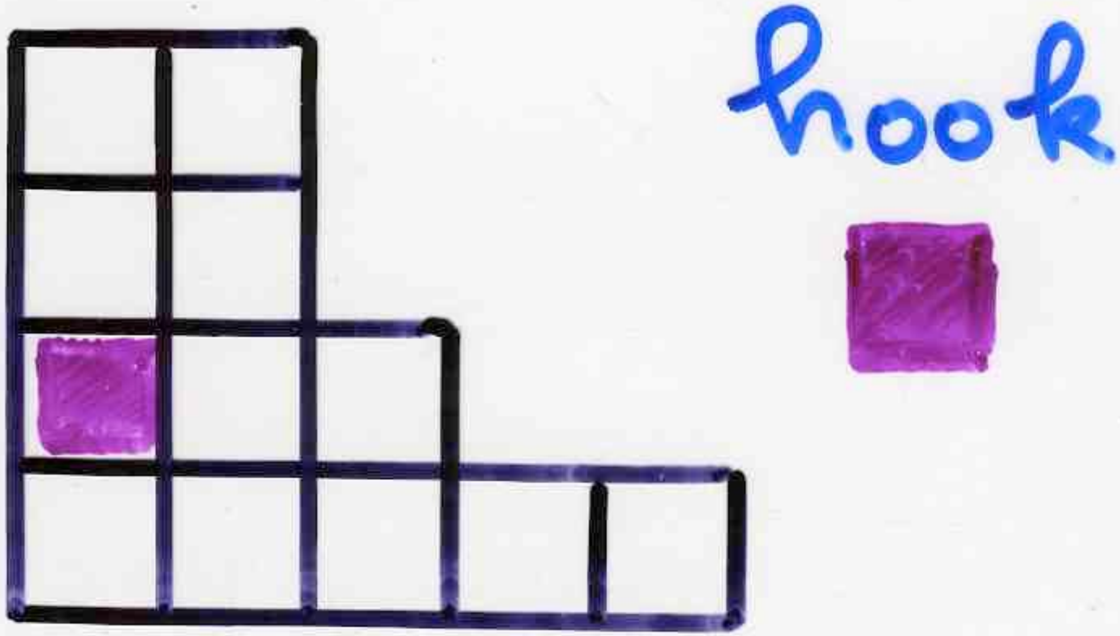
Young

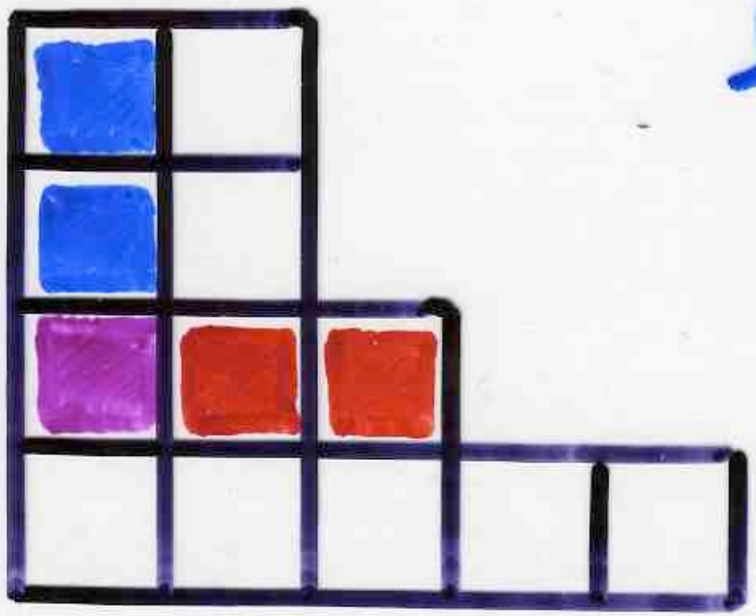
tableaux

shape λ

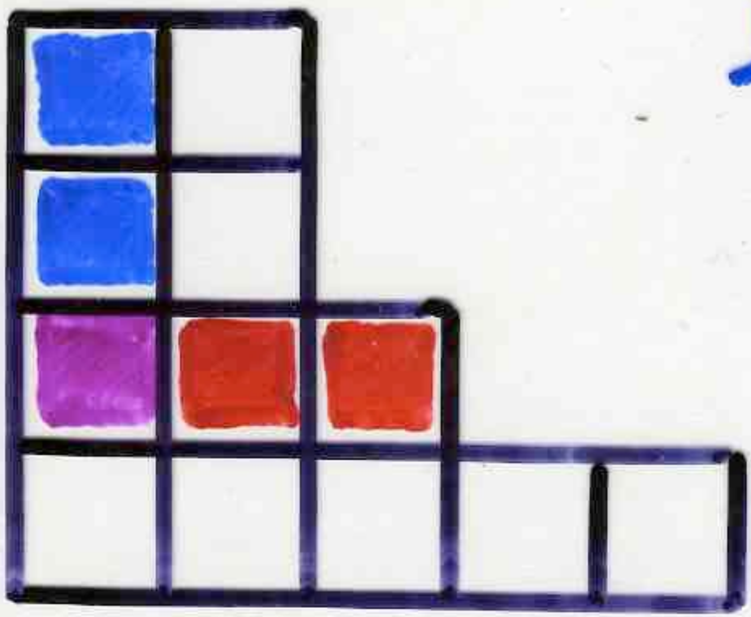
Hook length formula

J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954


hook



hook



hook length

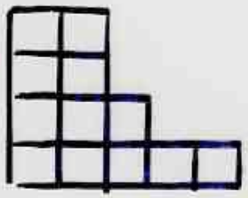
5

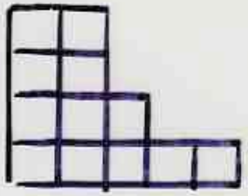
2	1			
3	2			
5	4	1		
8	7	4	2	1

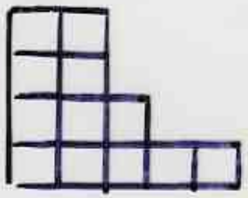
2	1			
3	2			
5	4	1		
8	7	4	2	1

$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula

ℓ  $=$

f  $= \dfrac{1\cdot 2\times 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{1^{3}\times 2^{3}\times 3\times 4^{2}\cdot 5\cdot 7\cdot 8}$

f  $= \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{1^{3}\cdot 2^{3}\cdot 3\cdot 4^{2}\cdot 5\cdot 7\cdot 8}$

$= 3^{4}\times 5\times 11 = 4455$

another example with binary trees

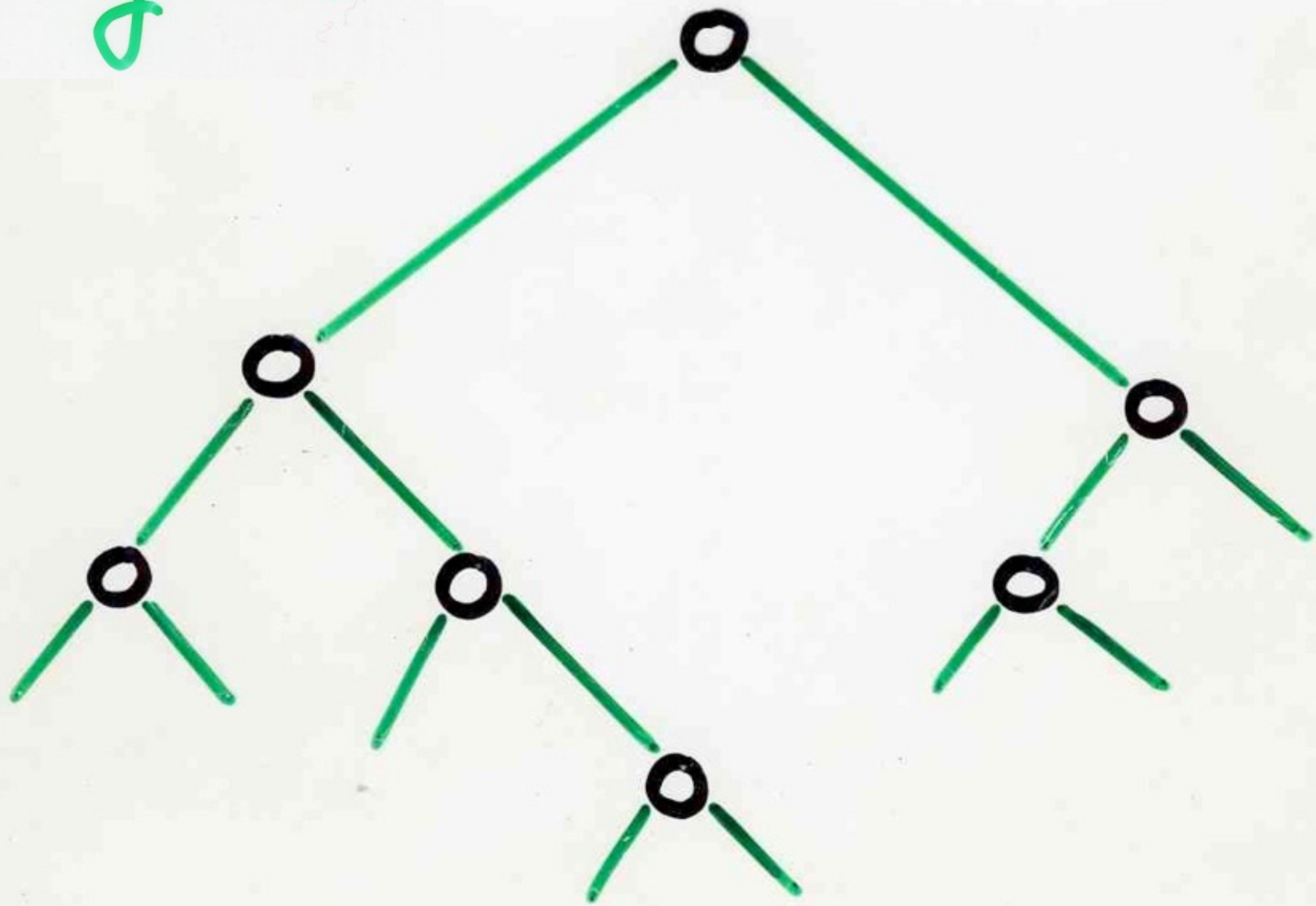
the use of

ordinary generating functions

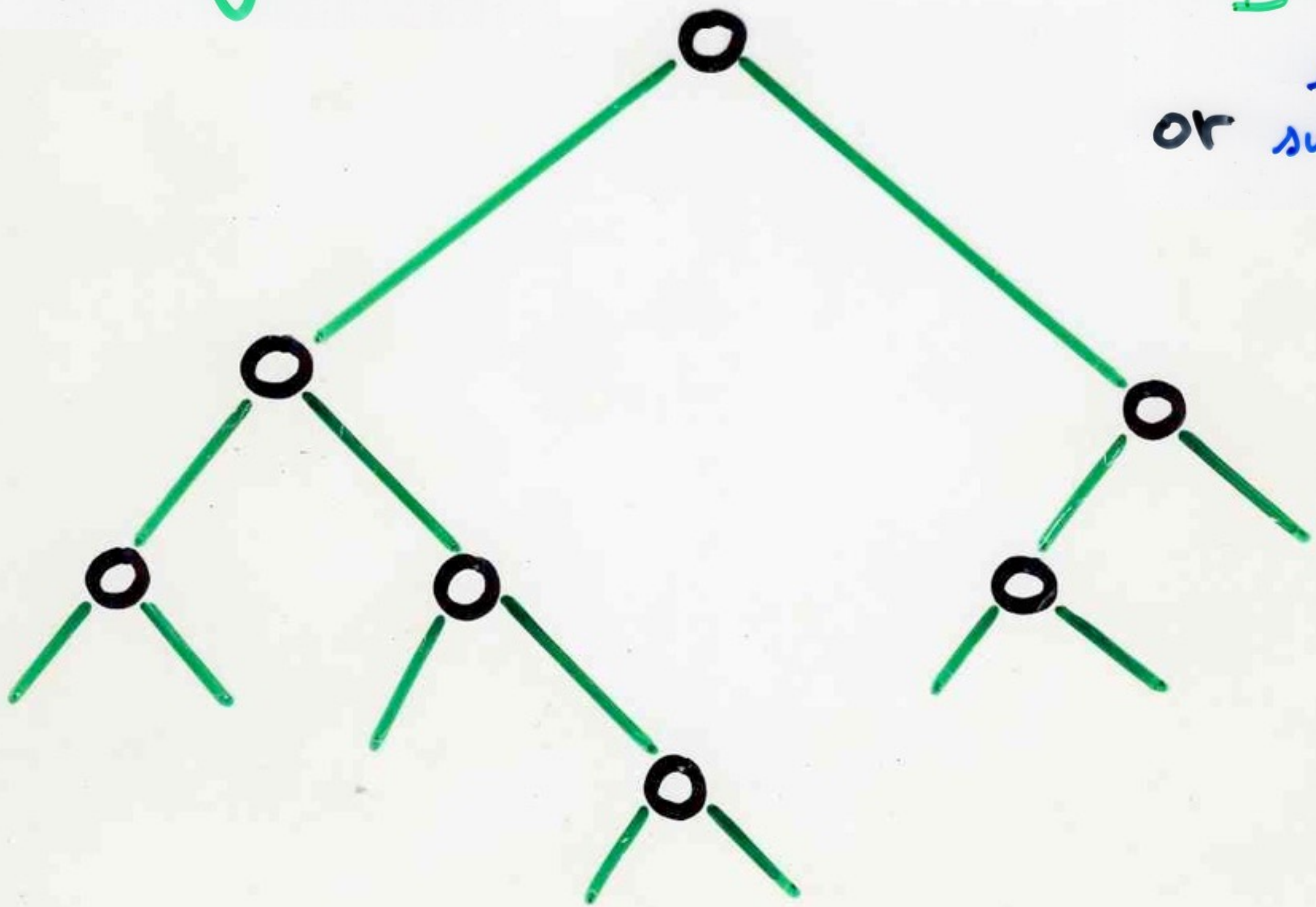
formal power series

(chapter 1)

binary tree



binary tree



$B = \langle L, r, R \rangle$

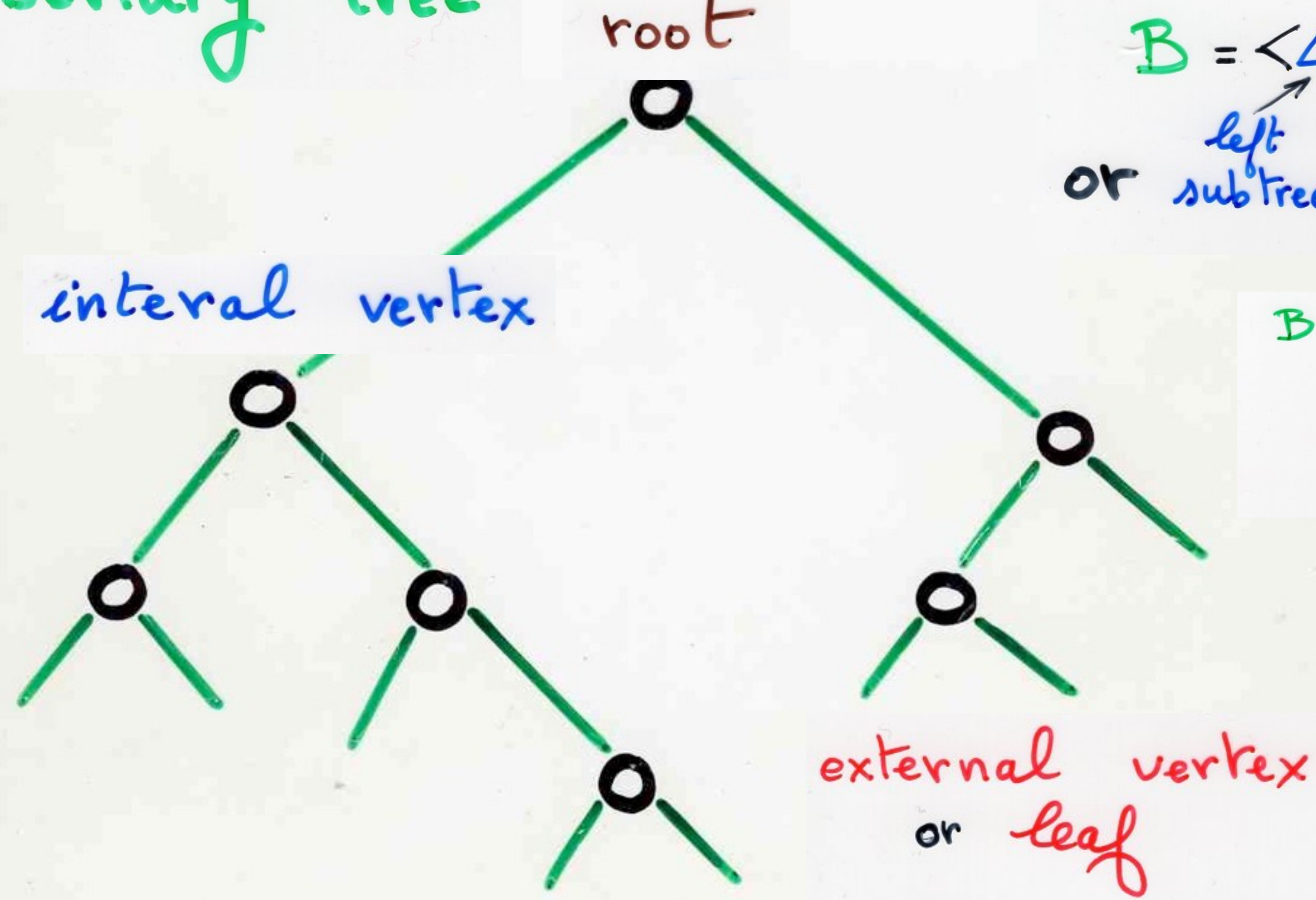
left subtree root right subtree

or

$B = \langle v \rangle$

leaf or external vertex

binary tree



$B = \langle L, r, R \rangle$

left subtree — root — right subtree

or

$B = \langle v \rangle$

leaf or external vertex

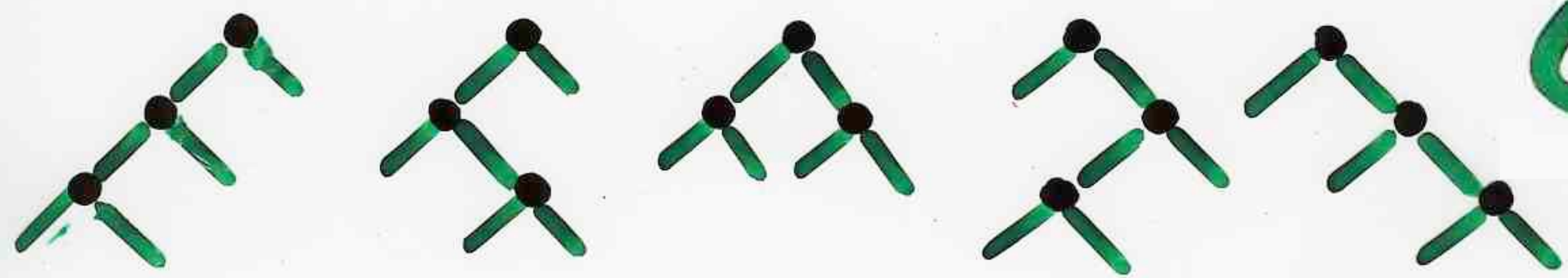
C_n = number of binary trees

having n internal vertices

(or $n+1$ leaves
= external vertices)

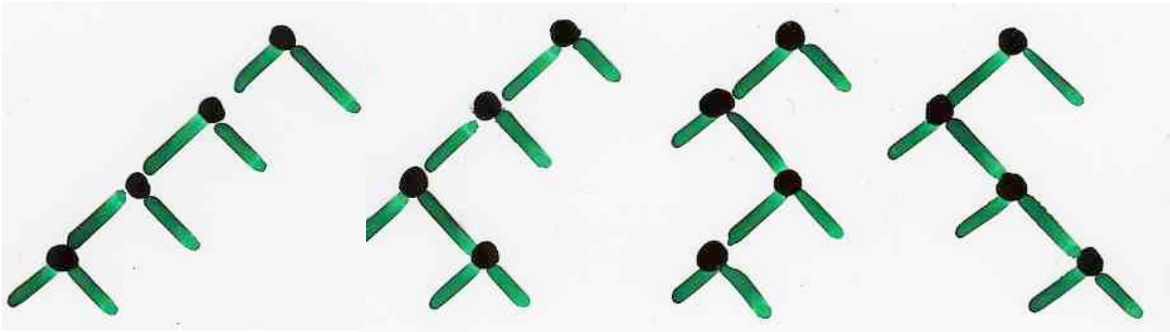
$C_1 = 1$

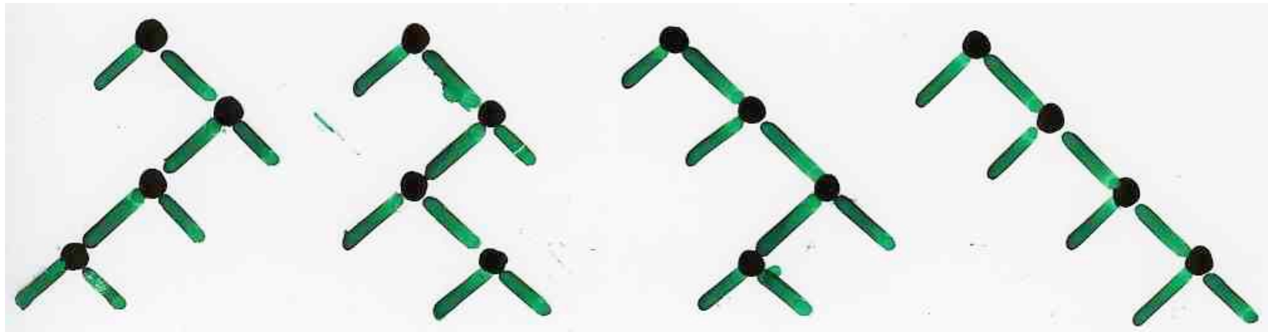
$C_2 = 2$

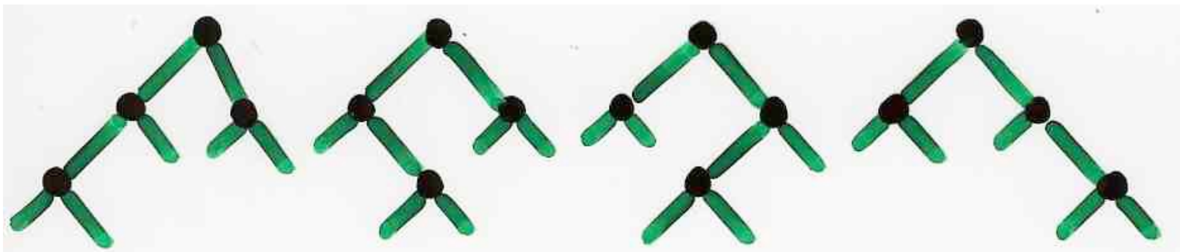


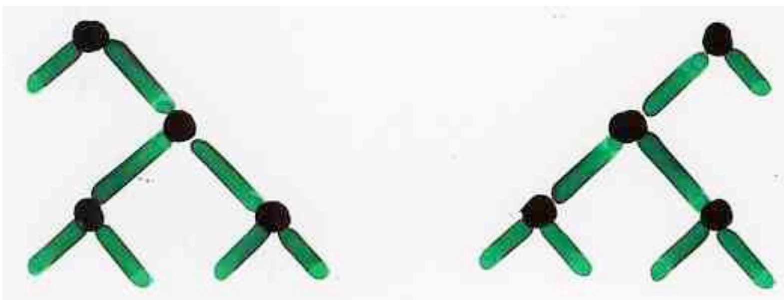
$C_3 = 5$

$C_4 = 14$









recurrence

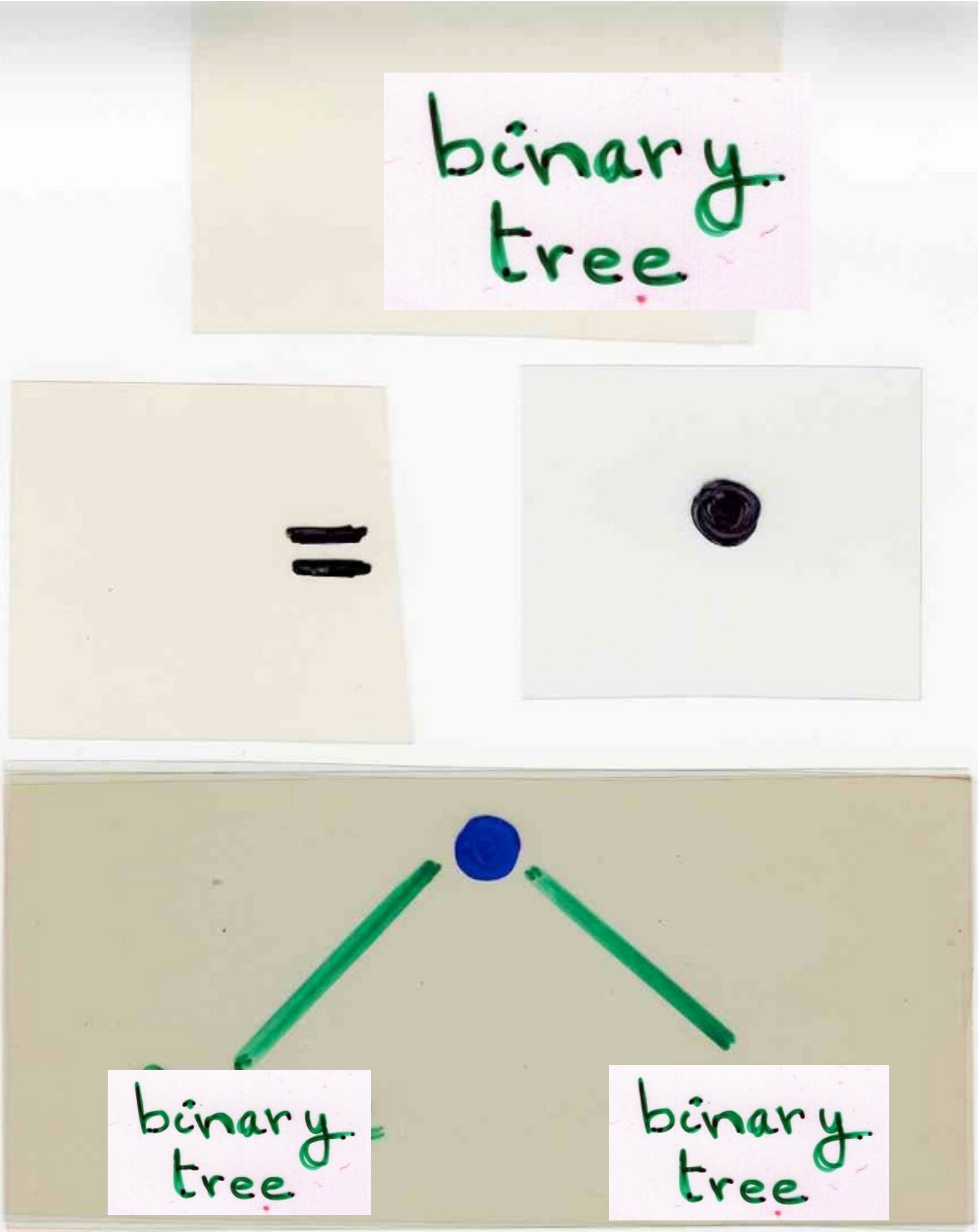
$$C_{n+1} = \sum_{i+j=n} C_i \, C_j$$

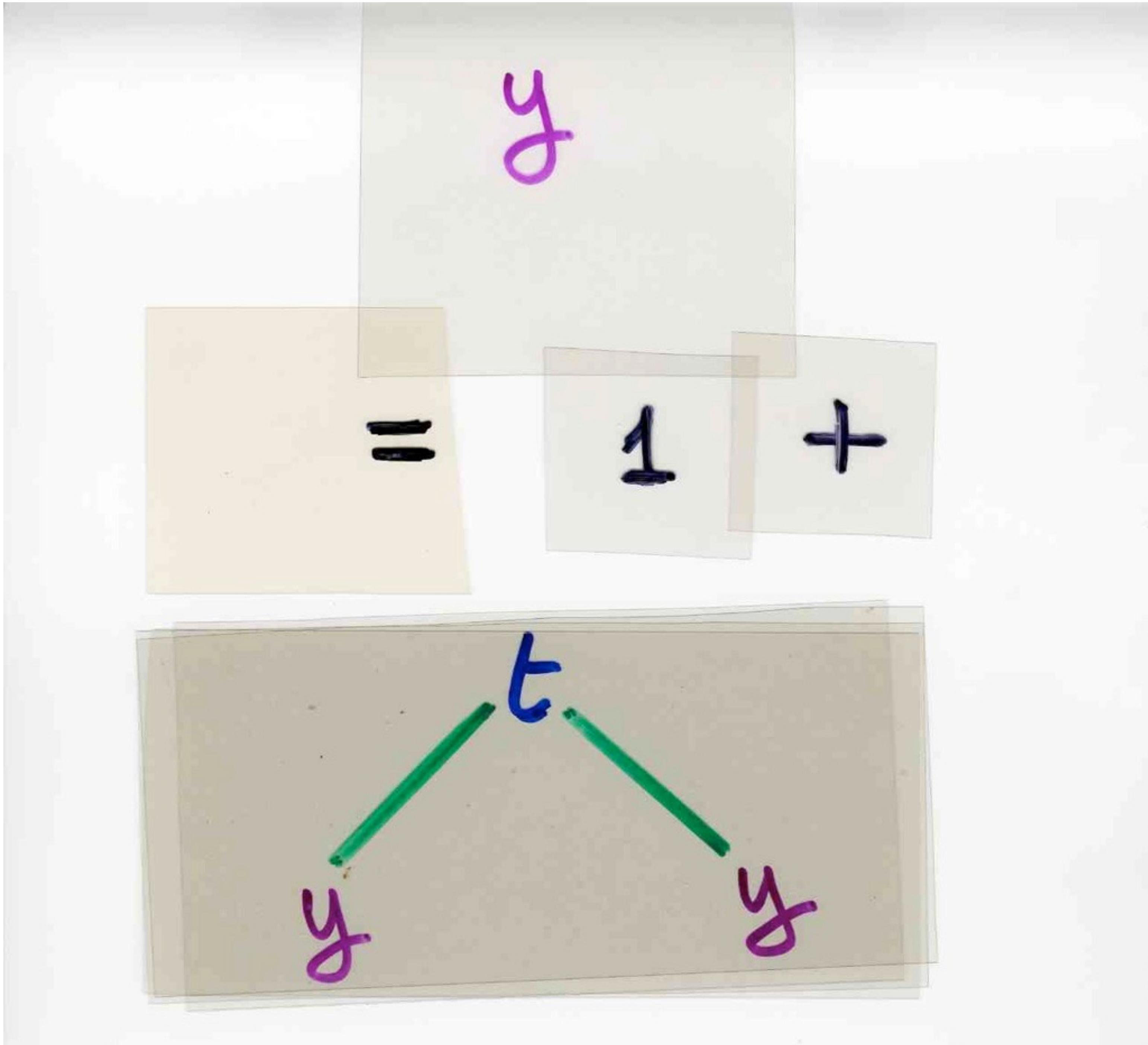
$$C_0 = 1$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\,n!}$$

$$n! = 1 \times 2 \times \dots \times n$$


binary tree
=
binary tree
binary tree


y
=
1
+
t
y
y

$$y = 1 + t(y)^2$$

$$y = 1 + t y^2$$

algebraic equation

$$y = 1 + 2t + 5t^2 + 14t^3$$
$$+ 42t^4 + \ldots + C_n t^n$$
$$+ \ldots$$

generating function

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots\ldots$$

$$\cdots + a_n t^n + \ldots\ldots$$

$$y = 1 + t\,y^2$$

algebraic equation

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\; n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

an example with alternating permutations

the use of exponential generating function

(chapter 2)

a famous sequence of numbers ...

$$1t + 2\frac{t^3}{3!} + 16\frac{t^5}{5!} + 272\frac{t^7}{7!} + \ldots$$

$$y = \tan t$$

tangent

D. André ($\simeq$ 1880)

alternating permutations

6 2 9 7 8 4 5 1 3

D. André ($\simeq$ 1880)

alternating permutations

$$6 \diagdown 2 \diagup 9 \diagdown 7 \diagup 8 \diagdown 4 \diagup 5 \diagdown 1 \diagup 3$$

$$T_{2n+1} = \sum_{i+j=n-1} \binom{2n+1}{2i-1} T_{2i-1} \, T_{2j-1}$$

$y = \tan t$

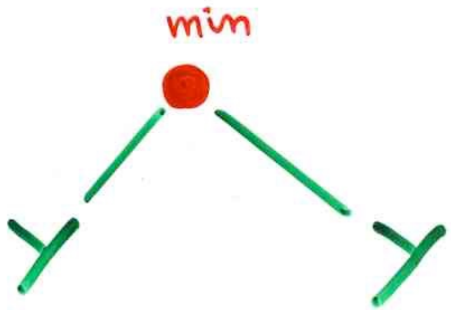
tangent

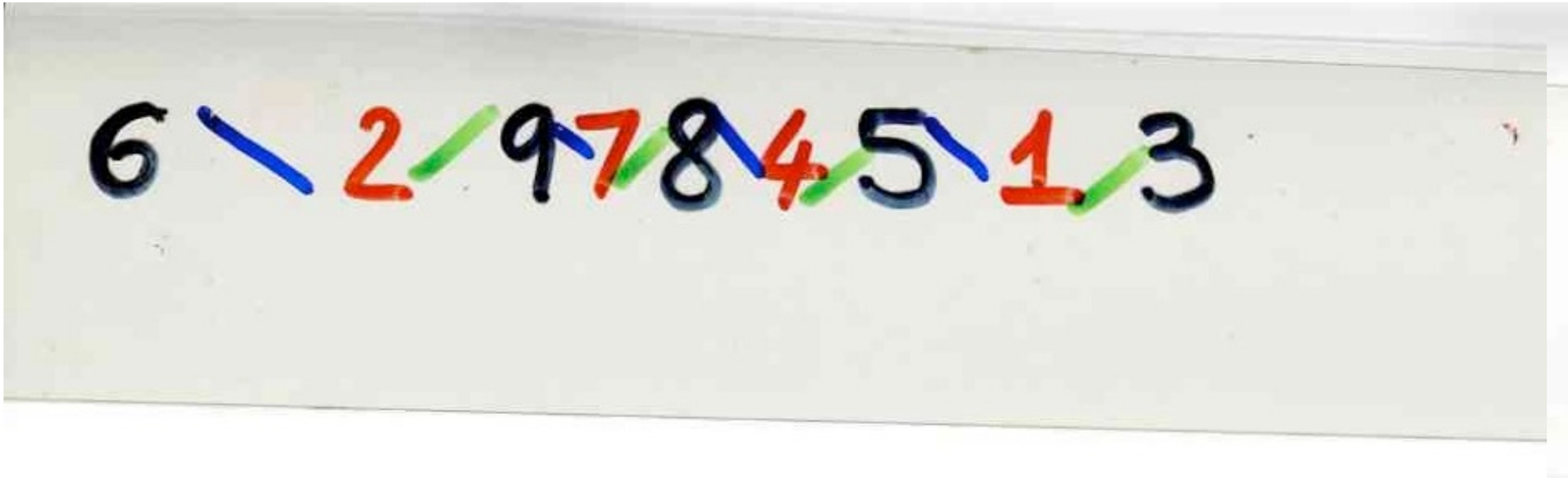
$$y = \sum_{n \geq 0} a_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$y' = 1 + y^2 \; ; \quad y(0) = 0$$

$$y = t + \int_0^t y^2 \, dt$$

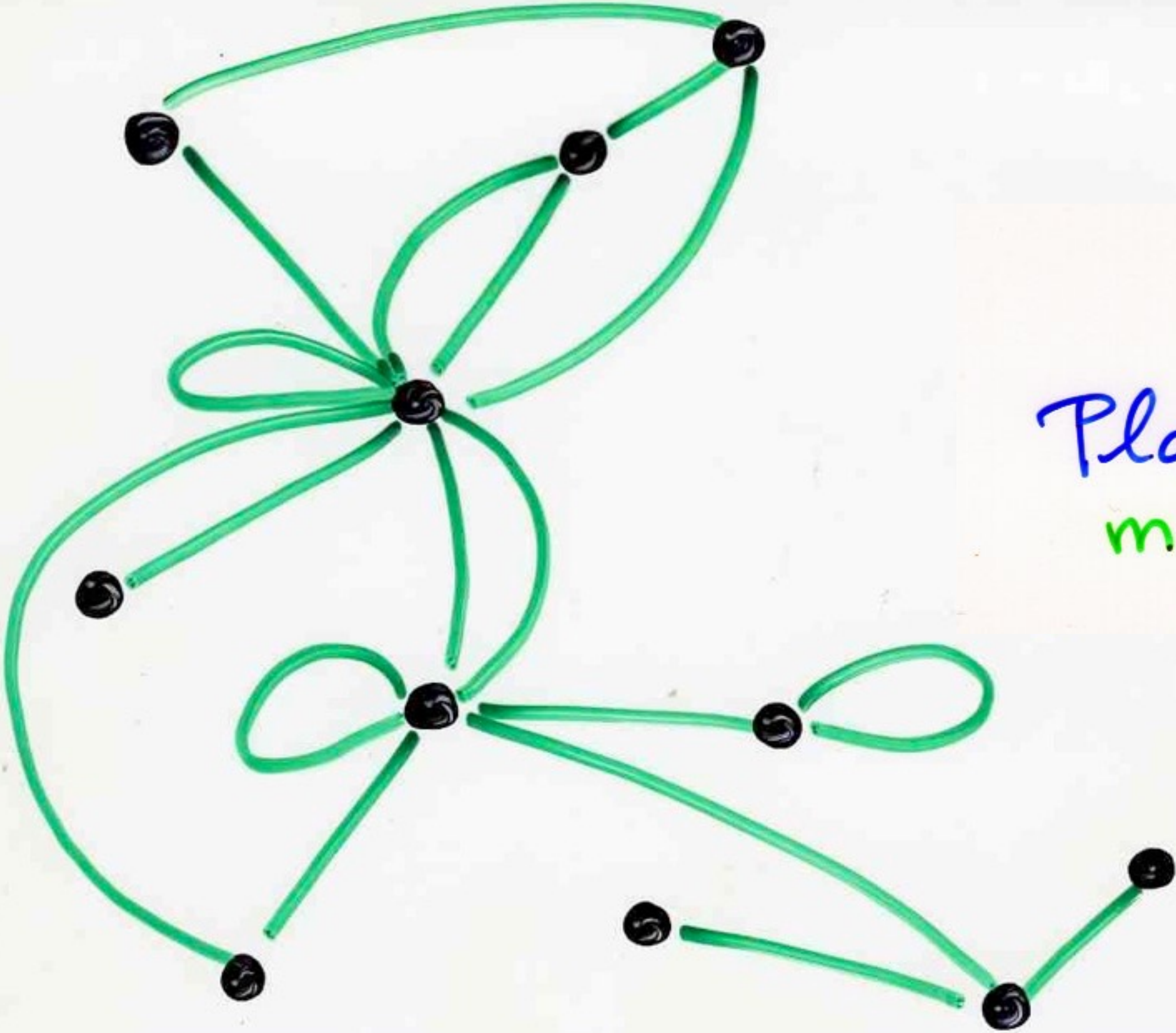
$$Y = T + \int_0^T Y^2(T) \, dT$$

$Y =$  ou 

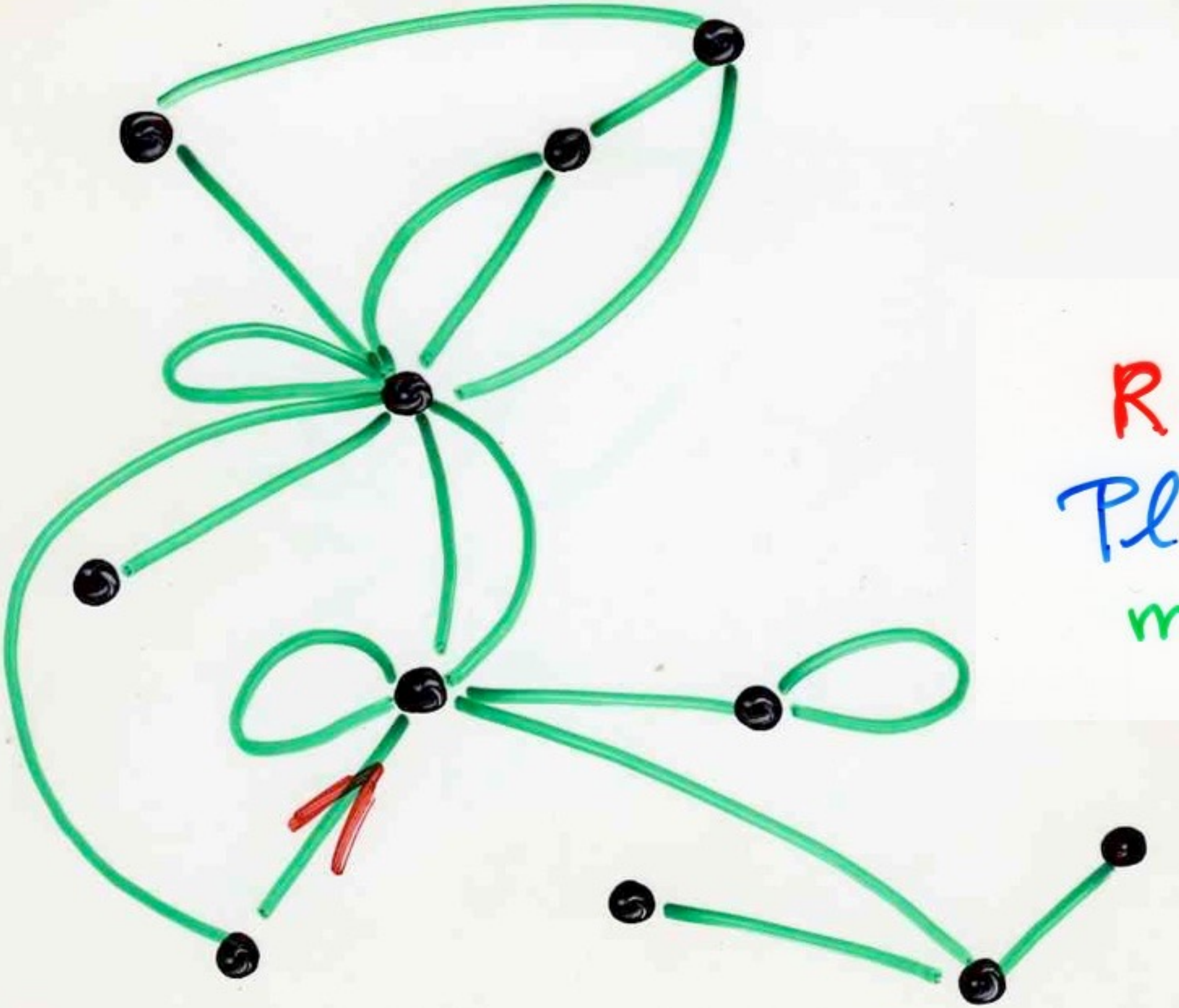

6 2 9 7 8 4 5 1 3

bijective combinatorics

example: planar maps



Planar
map



Rooted
Planar
map

Tutte (1960)

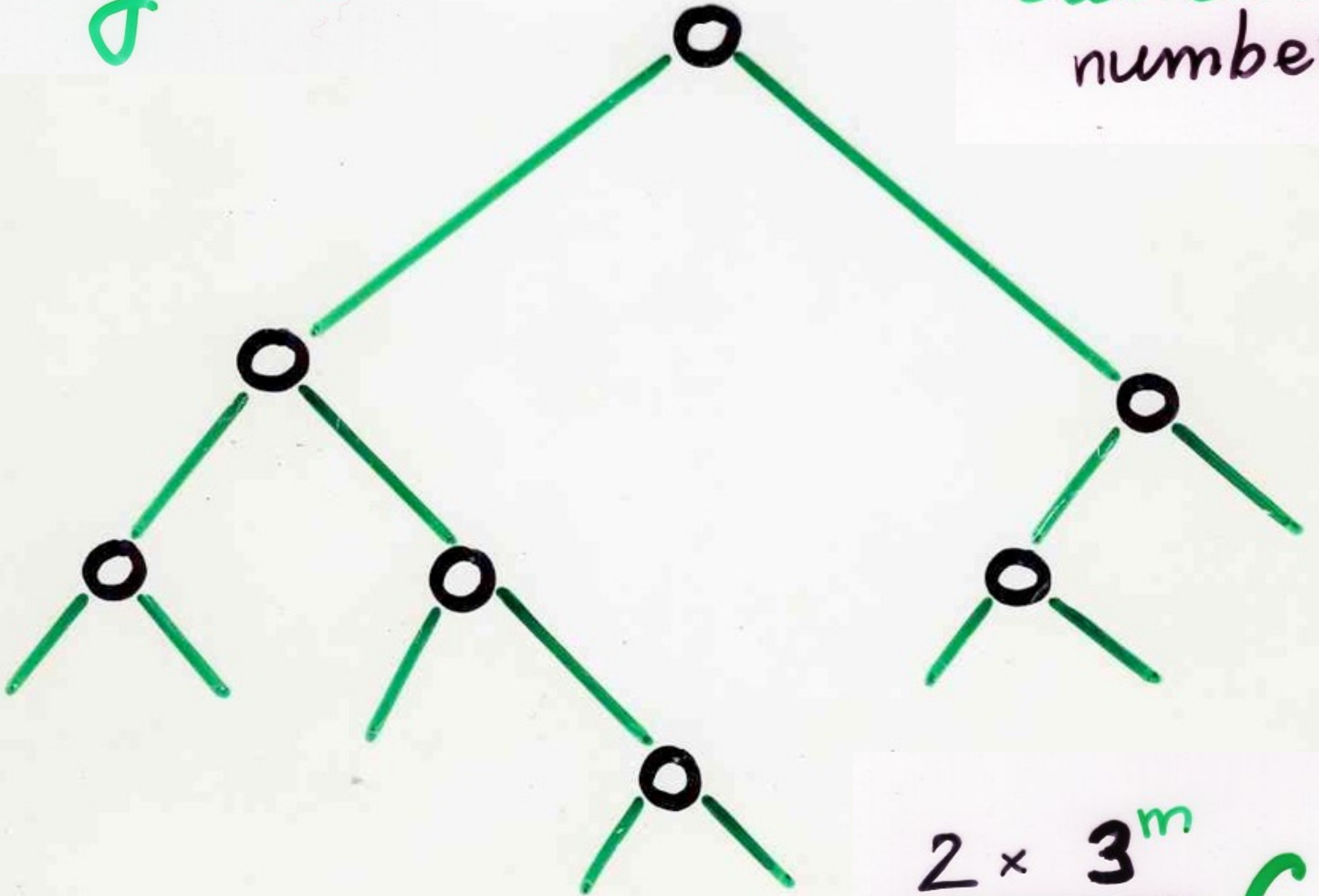
The number of rooted planar maps with m edges is

$$\frac{2 \times 3^m}{(m+2)} C_m$$

Catalan number

binary tree

Catalan number



$$\frac{2 \times 3^m}{(m+2)} C_m$$

Cori, Vauquelin (1970, ...)

Arquès (1980, ...)

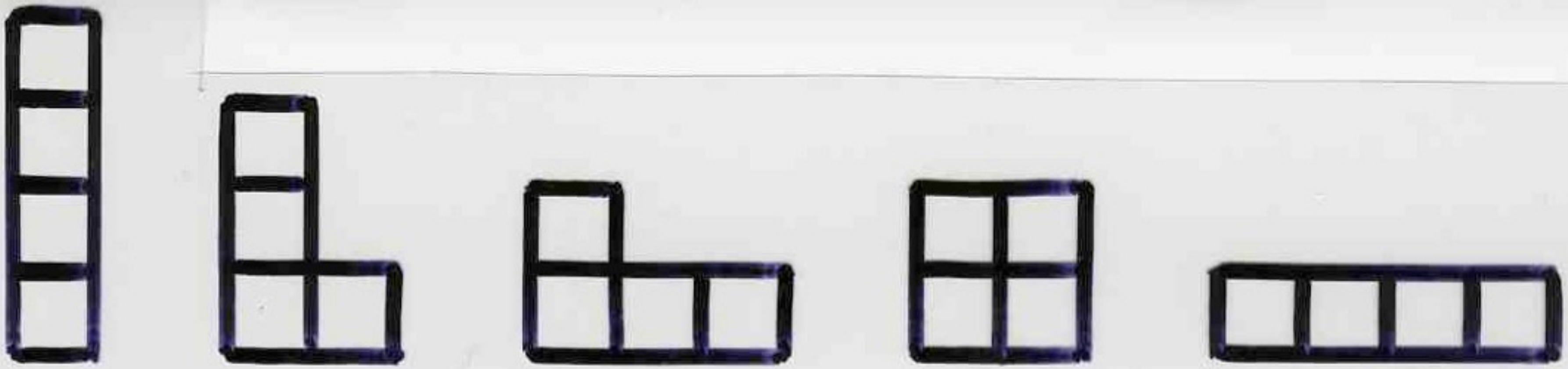
Schaeffer (1997, ..)

Boutier, Di Francesco, Guitter (2002, ...)

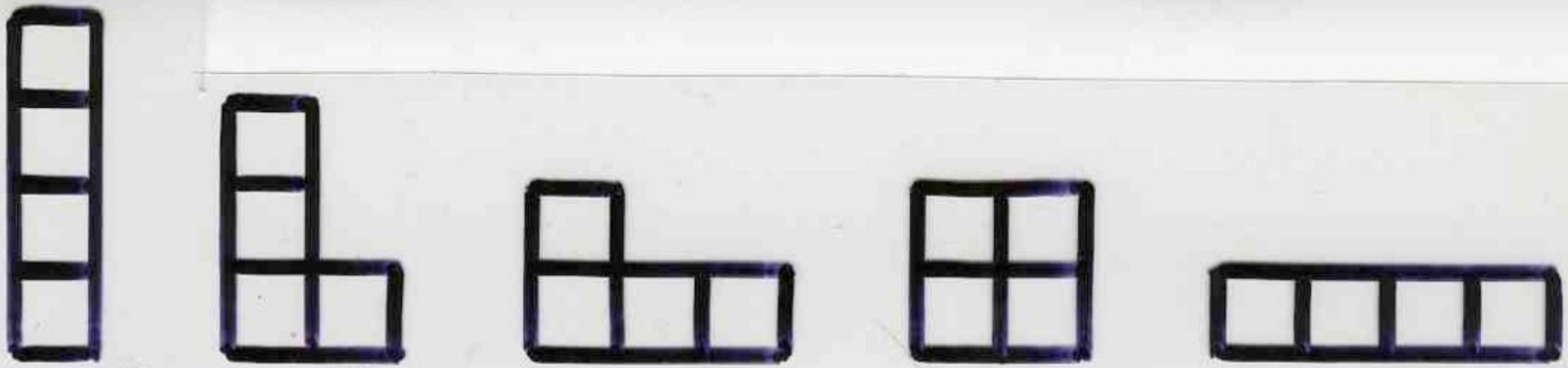
quantum
gravity

bijective proof of an identity

example: RSK



1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

$$n! = \sum_{\substack{\lambda \\ \text{partitions} \\ \text{of } n}} \left(f_\lambda \right)^2$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

6	10			
3	5	8		
1	2	4	7	9

P

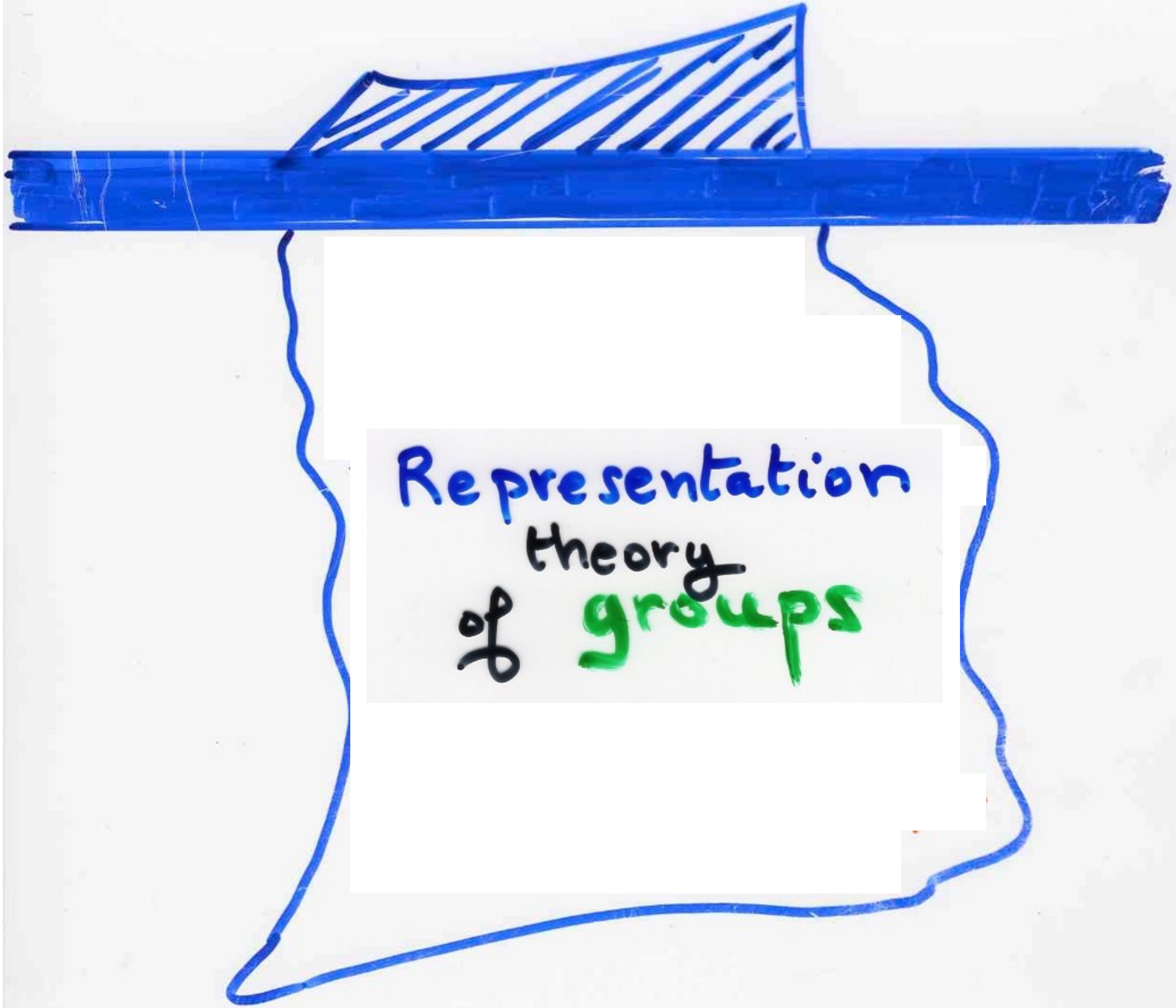
8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

The Robinson-Schensted correspondence

The Robinson-Schensted correspondence



algebraic combinatorics

Representation theory of groups

see a group G as a (sub)-group of matrices

$$G \to \text{Matrices}$$

$n \times n$, coeff. in $\mathbb{R}$

see G as a group of transformations

for every group

representation $\xrightarrow[\text{decomposition}]{}$ into irreducible representations

analogy [every number n $= p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

prime numbers decomposition

Case of the group $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) $= f_\lambda$

number of Young tableaux with shape λ

in (finite) group theory :

$$|G| = \sum_{R} (\deg R)^2$$

order of the group

R irreducible representation

for the symmetric group $\mathfrak{S}_n$:

$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ partitions of n

[redacted]

$=$

$\longleftrightarrow$

=

better
understanding

$\longleftrightarrow$

“The bijective paradigm”

"drawing" calculus ...

computing with "drawings"

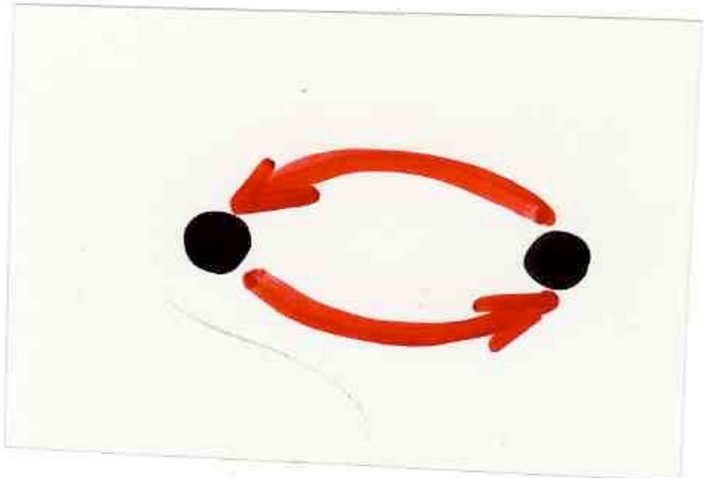
(figures)

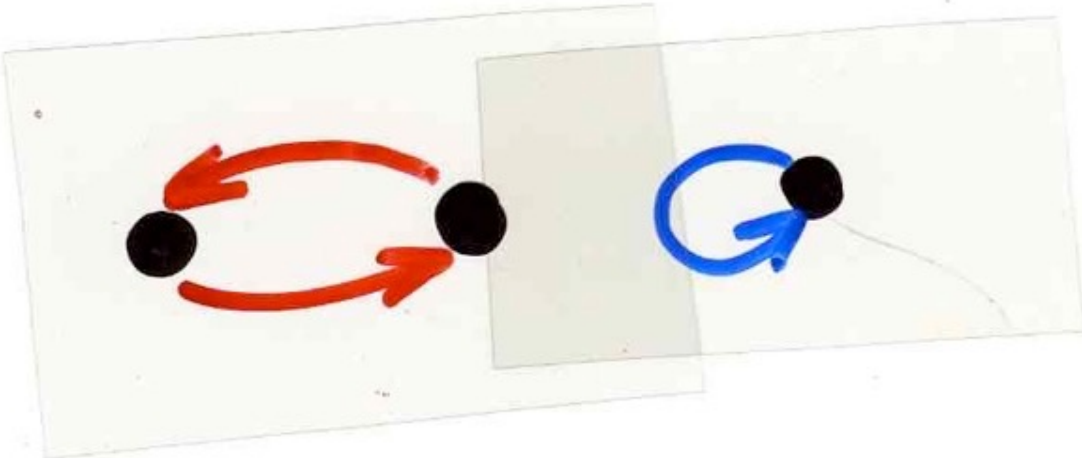
example

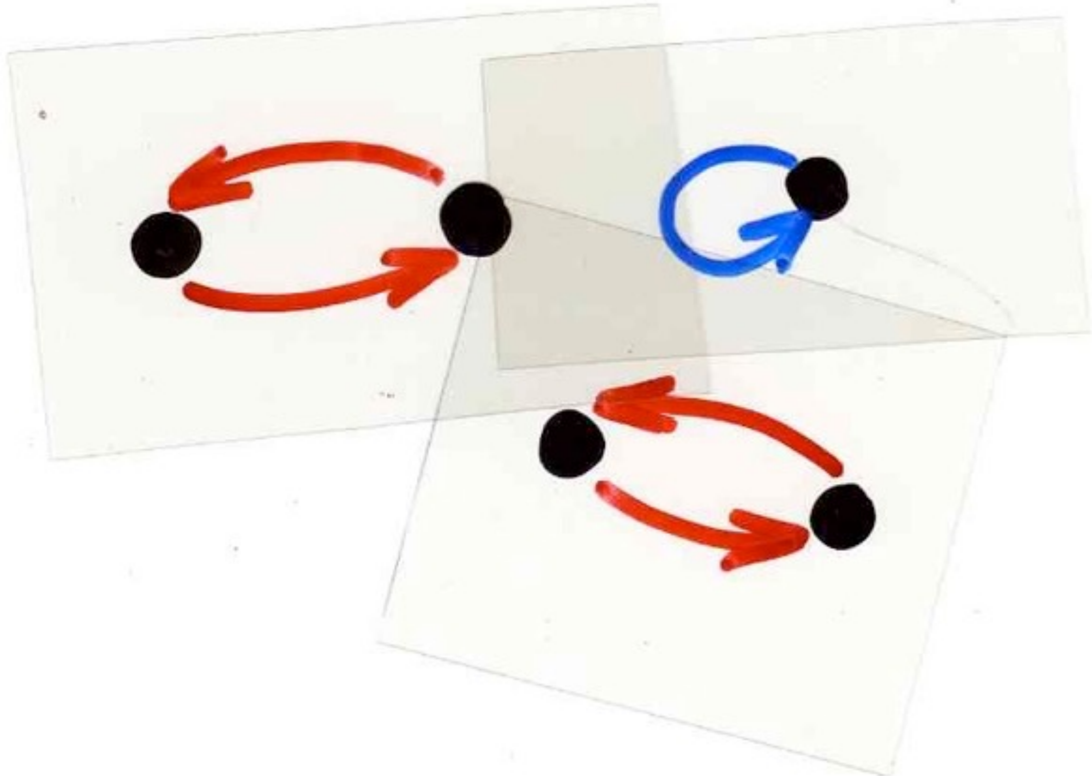
Mehler identity for
Hermite polyomials

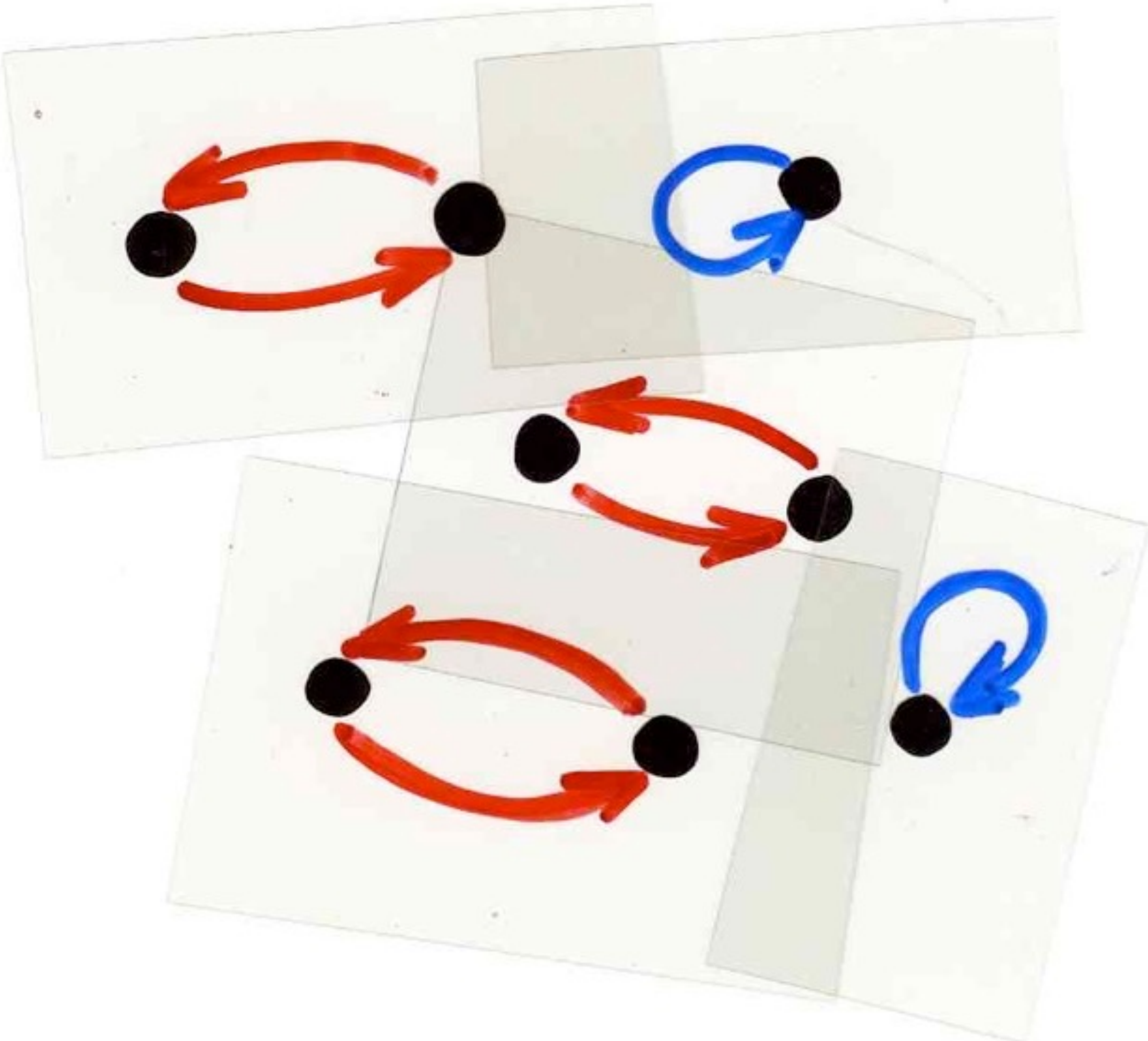
$$\sum_{n \geqslant 0} H_n(x) H(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

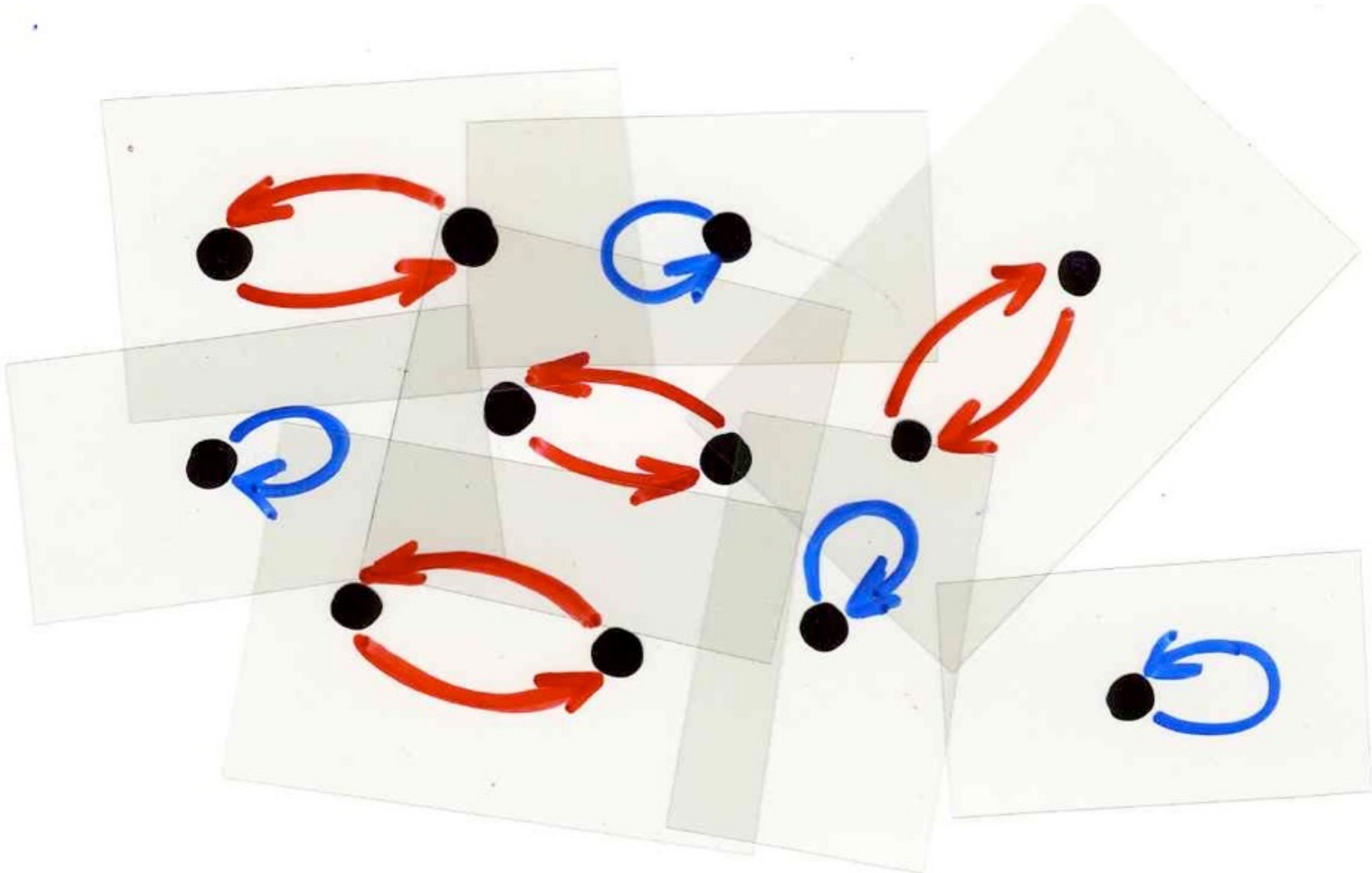
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(x t - \frac{t^2}{2}\right)$$

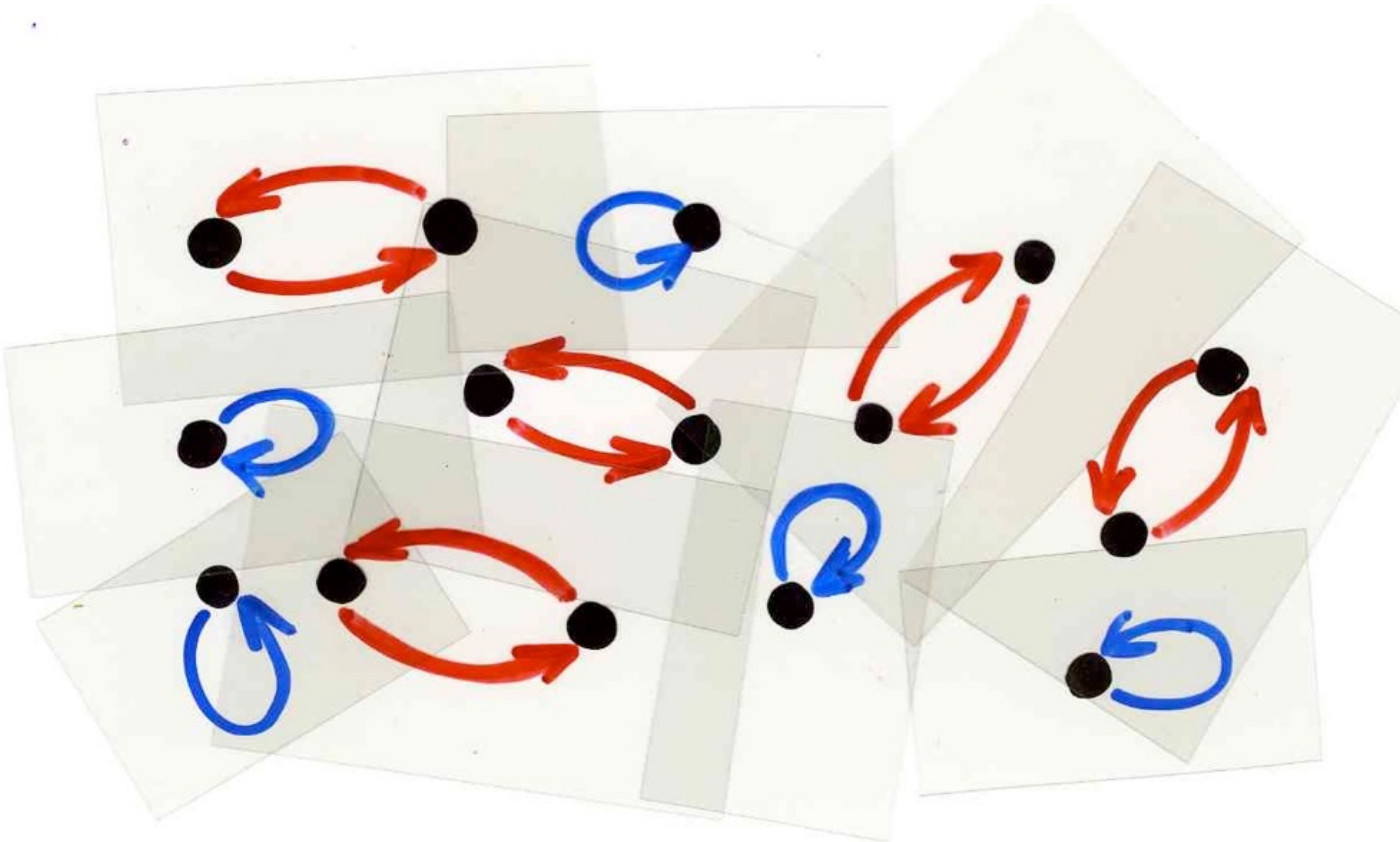




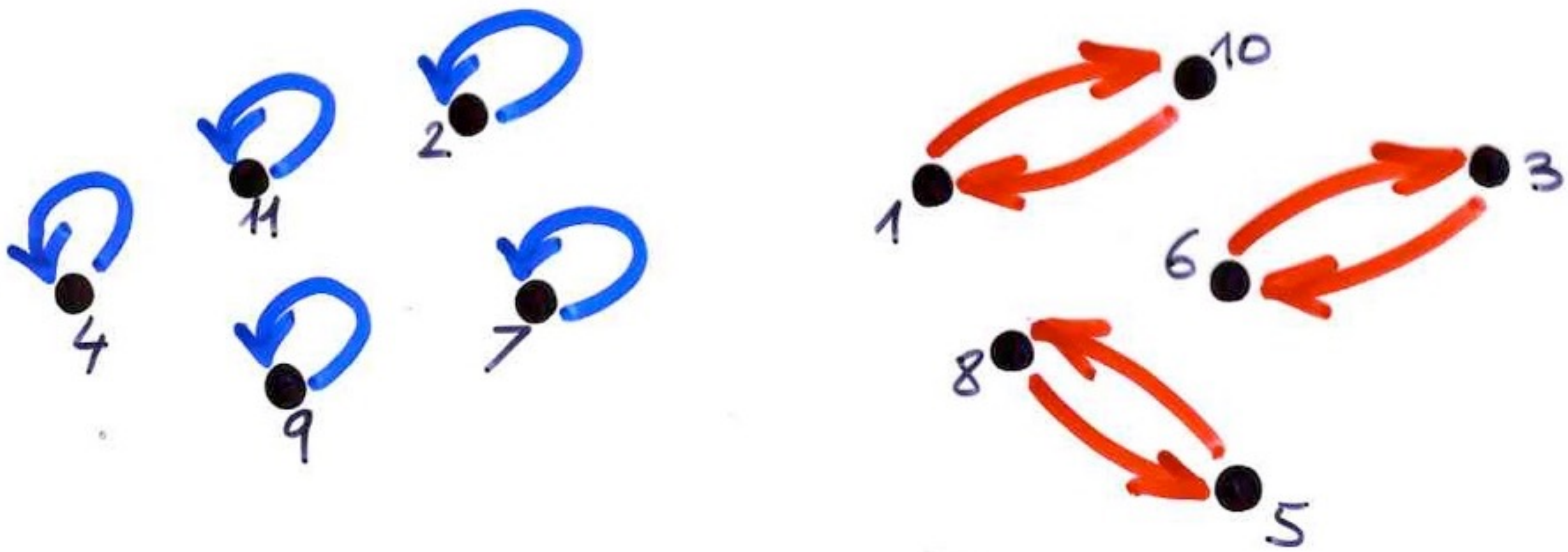




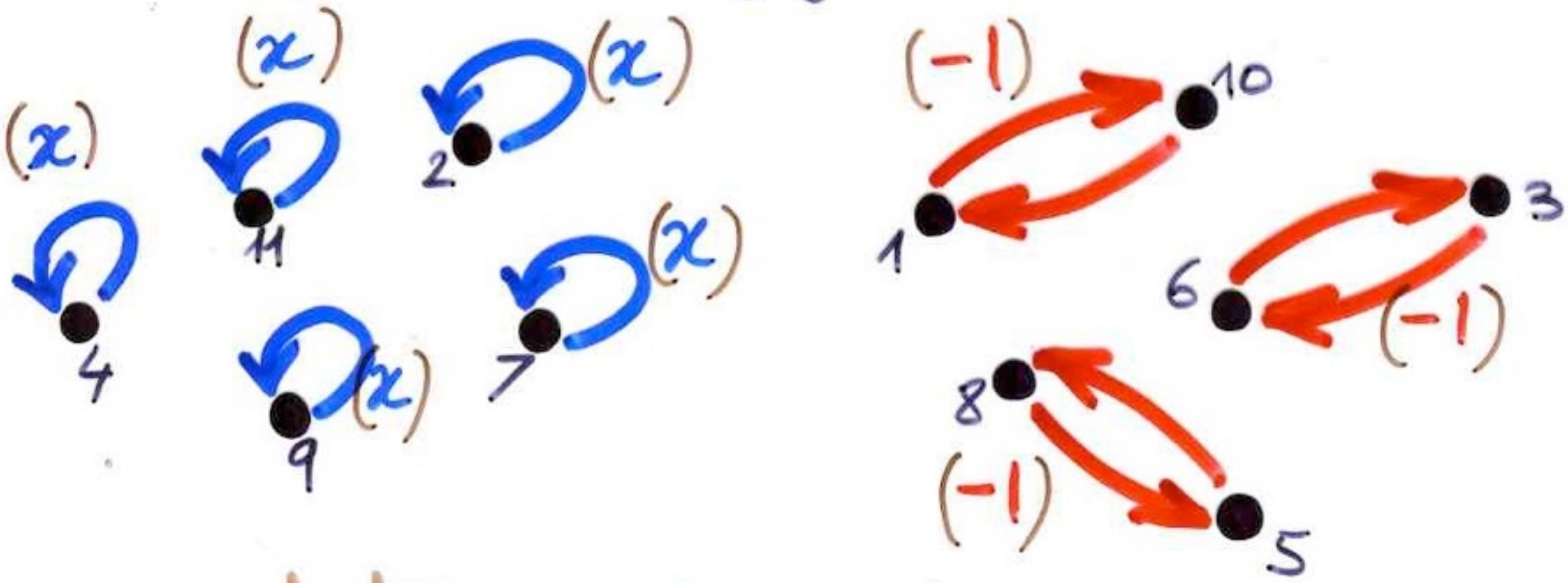




Hermite configurations



Hermite configurations



weight

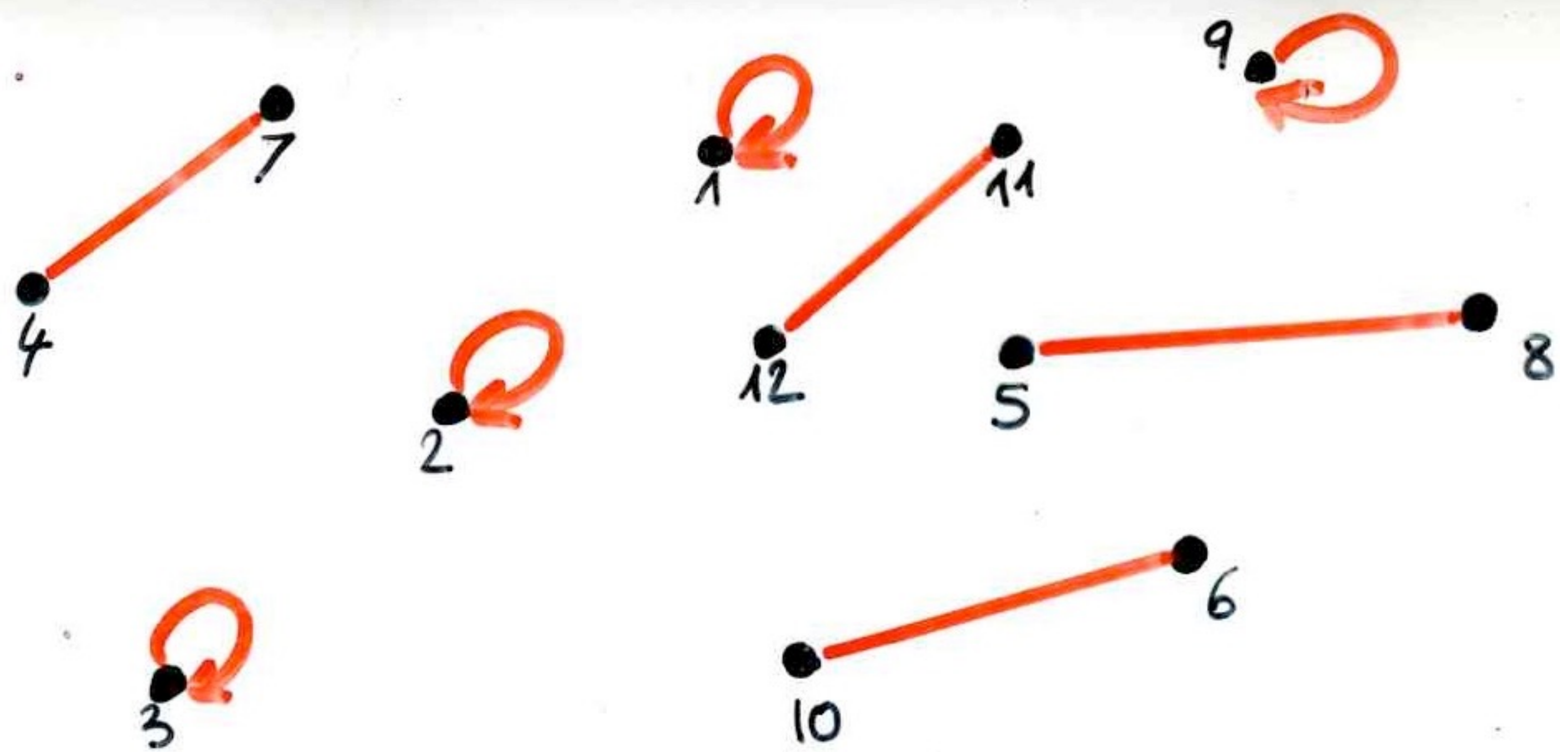
(x)

(-1)

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \quad \frac{t^n}{n!} =$$

$$H_n(y)$$


9
7
1
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4
2
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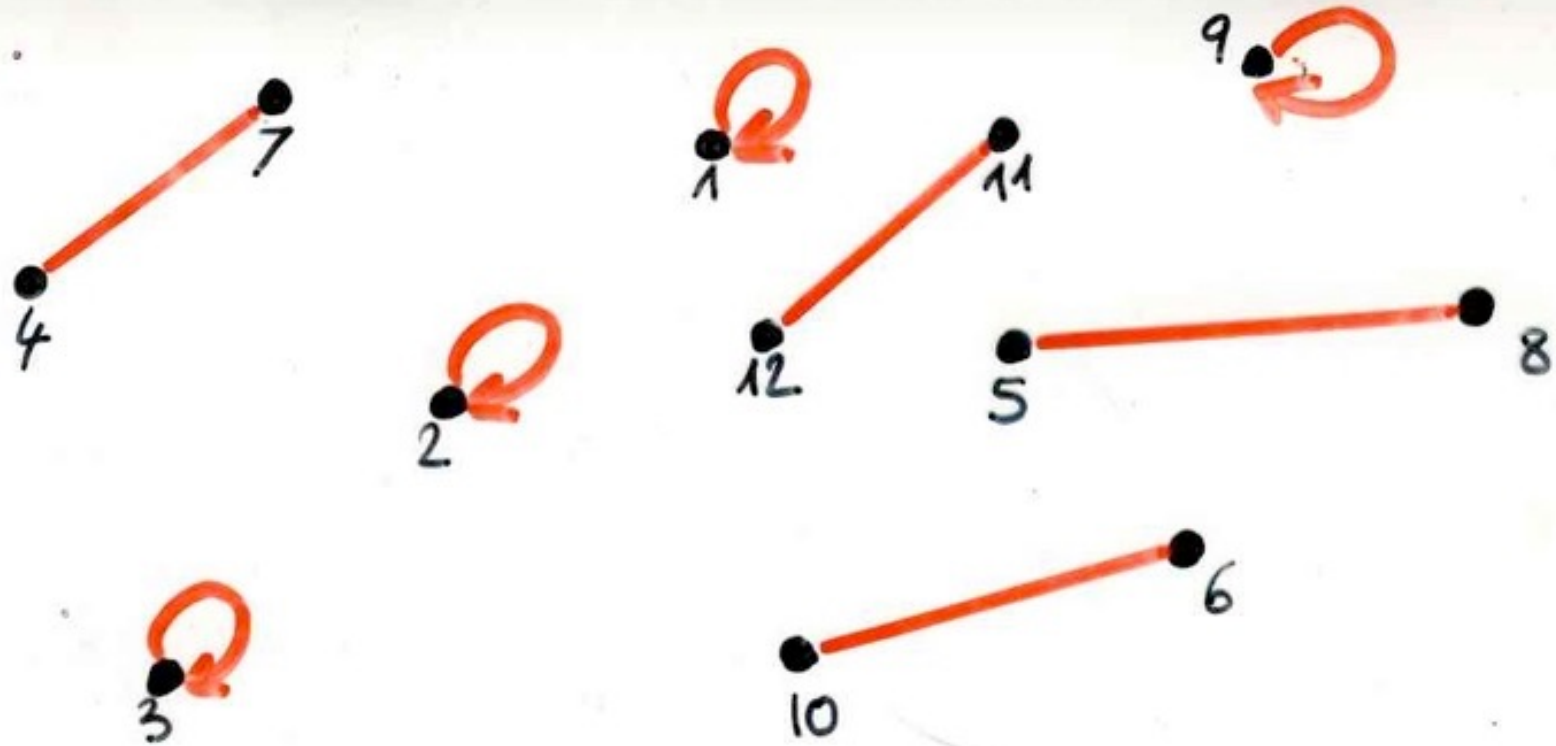
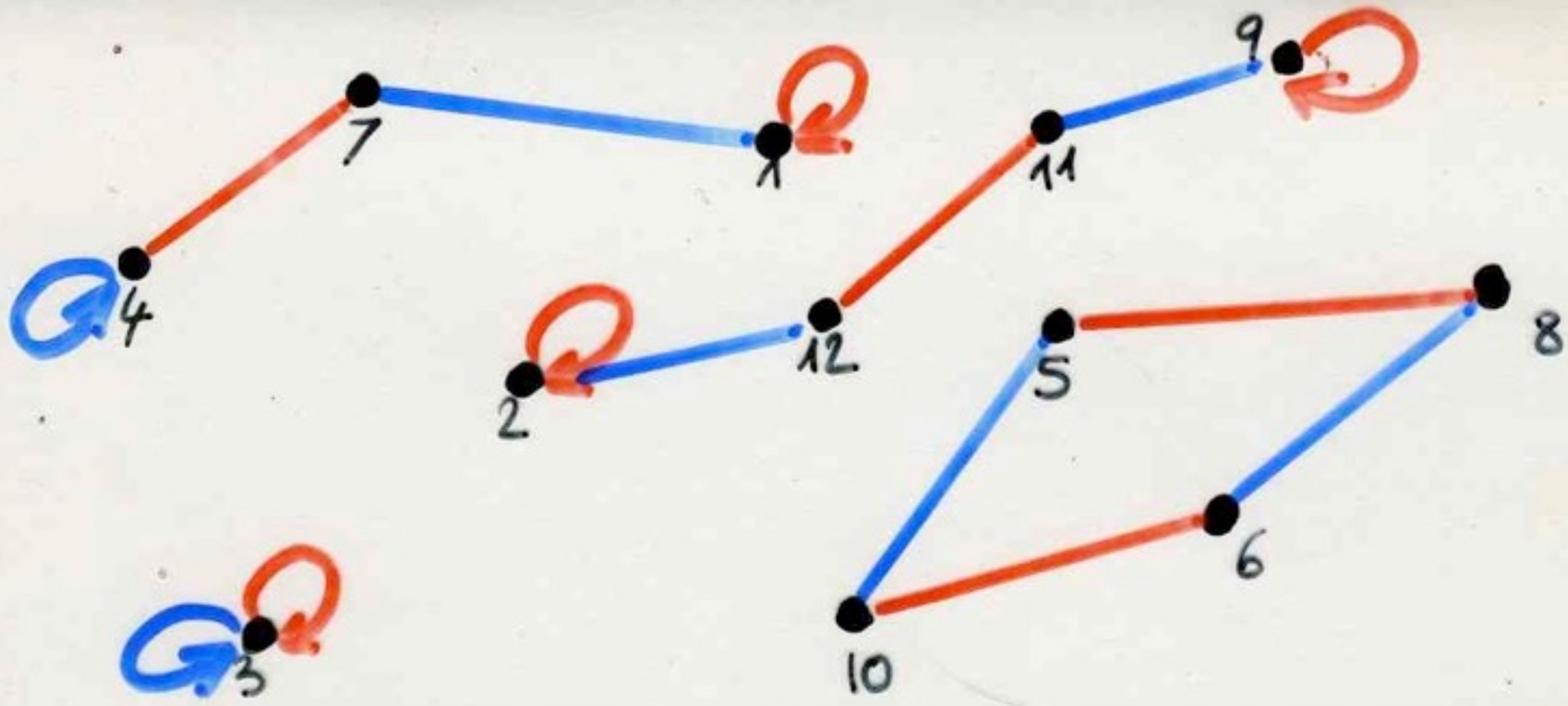
5
8
6
3
10



$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\;y^2\;t^2}{1-4t^2}$$

$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

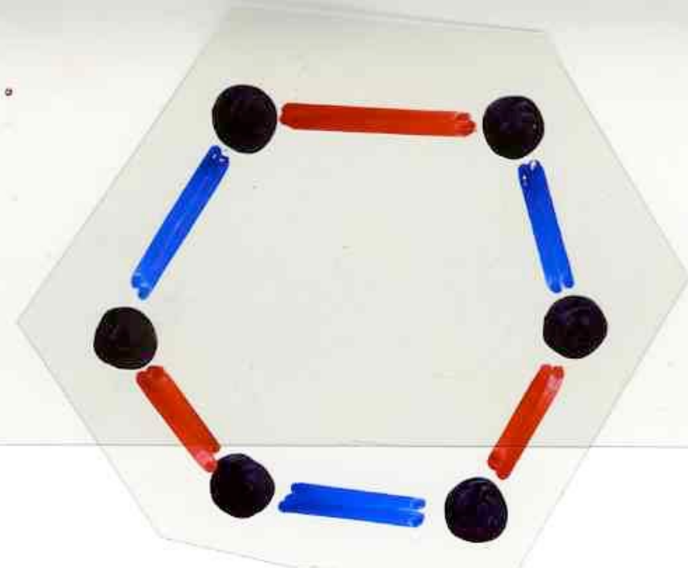
$$\frac{-4\;x^2\;t^2}{1-4t^2}$$

$$\frac{4xyt}{1-4t^2}$$

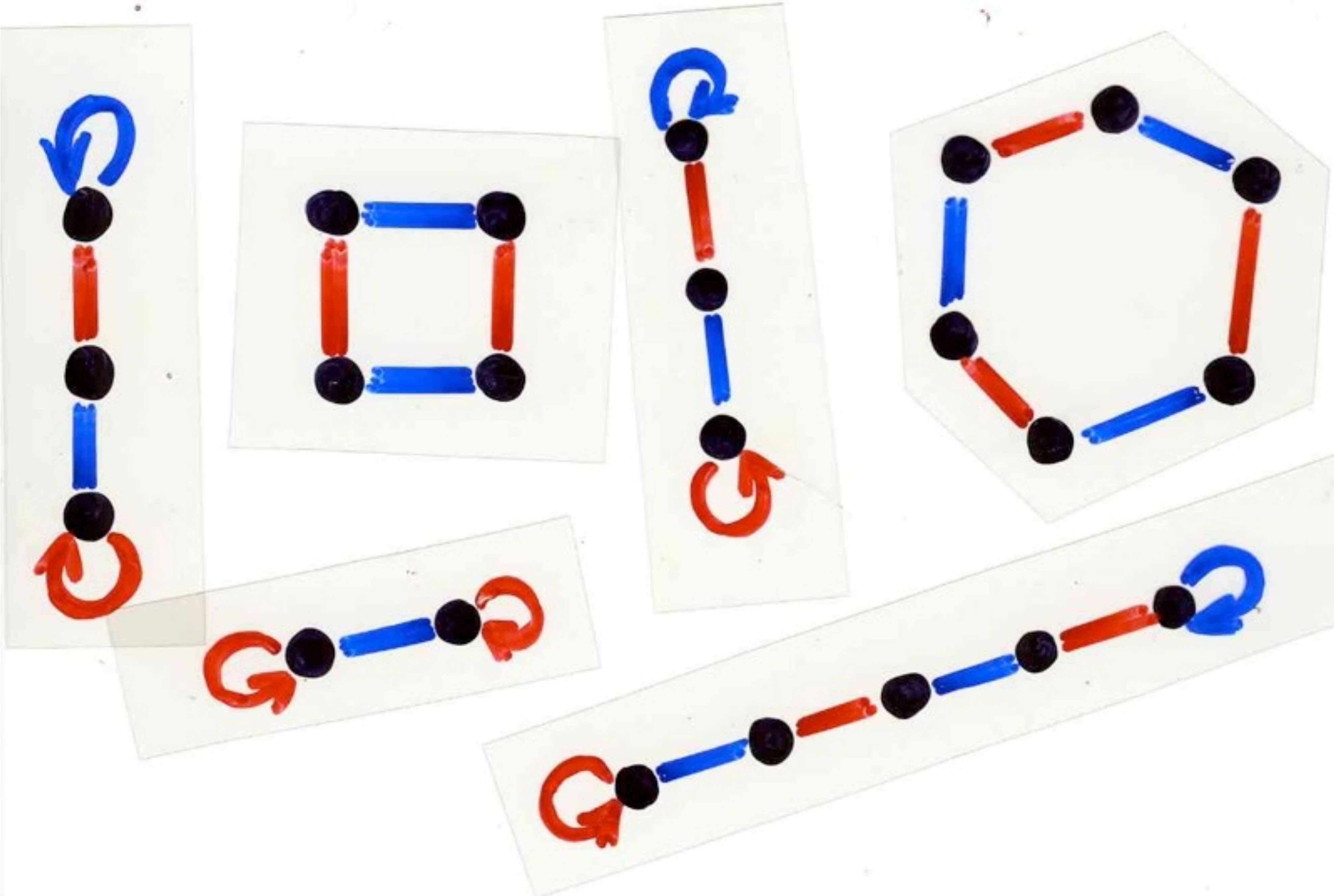
$$\frac{-4x^2t^2}{1-4t^2}$$

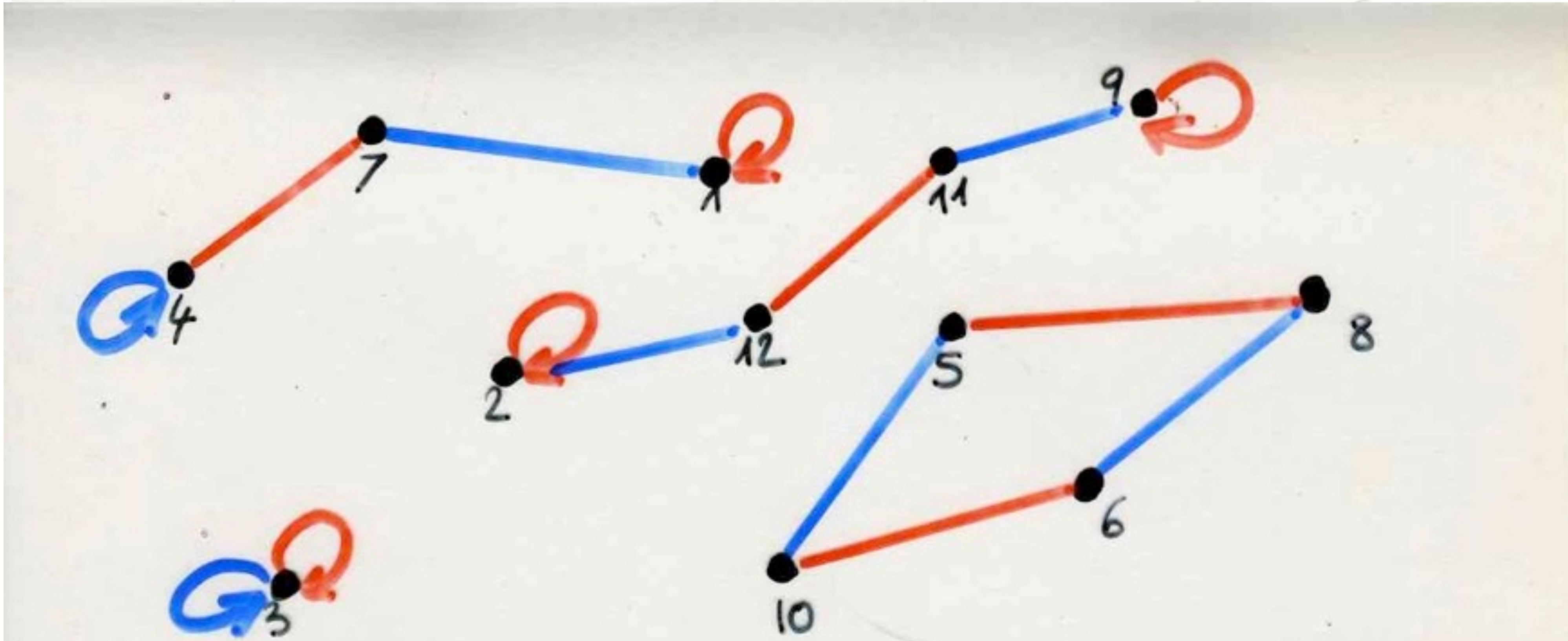
$$\frac{-4y^2t^2}{1-4t^2}$$

$$\frac{4xyt}{1-4t^2}$$



$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$





7
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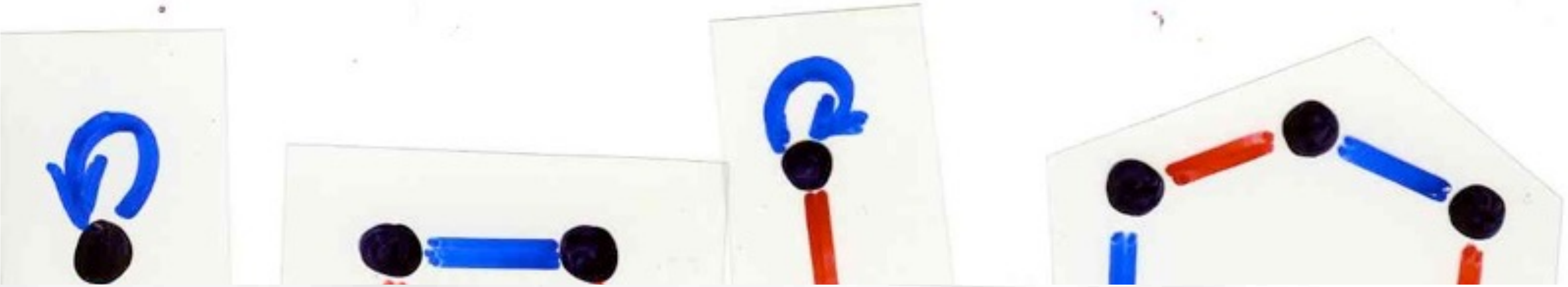
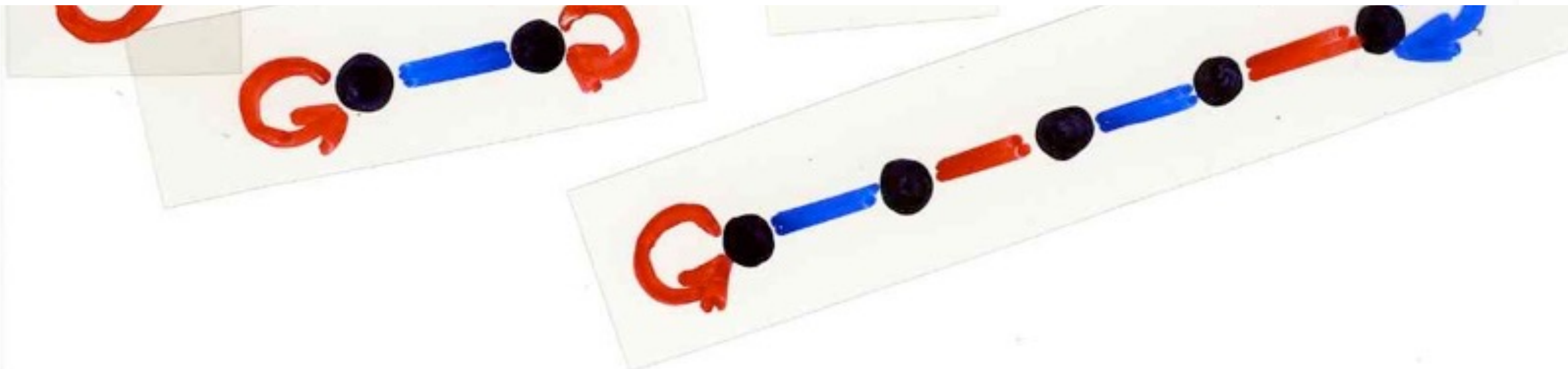
10



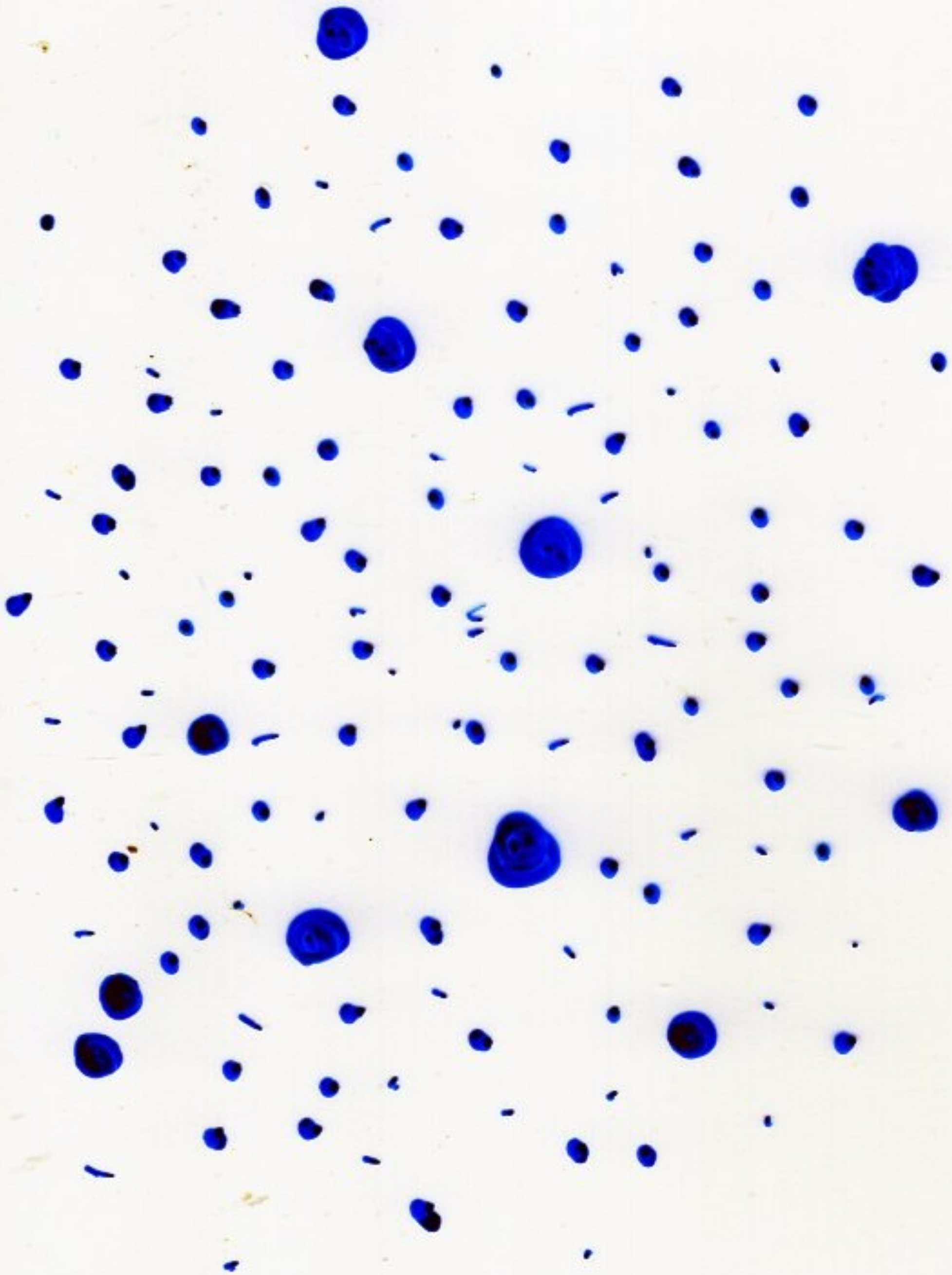
$$\sum_{n\geq 0} H_n(x)\, H(y)\, \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



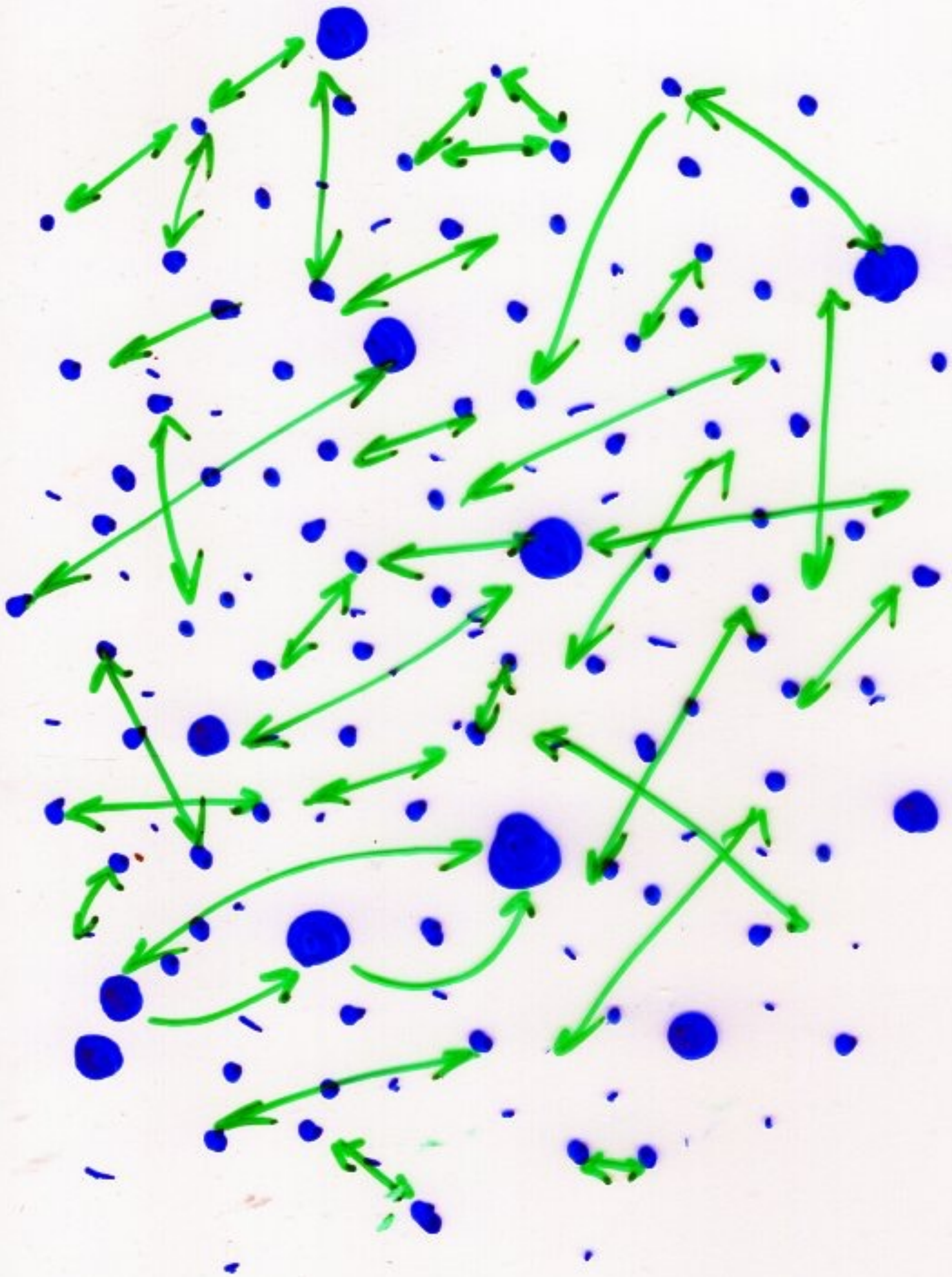
Identities

Bijections

"Bijective tools"

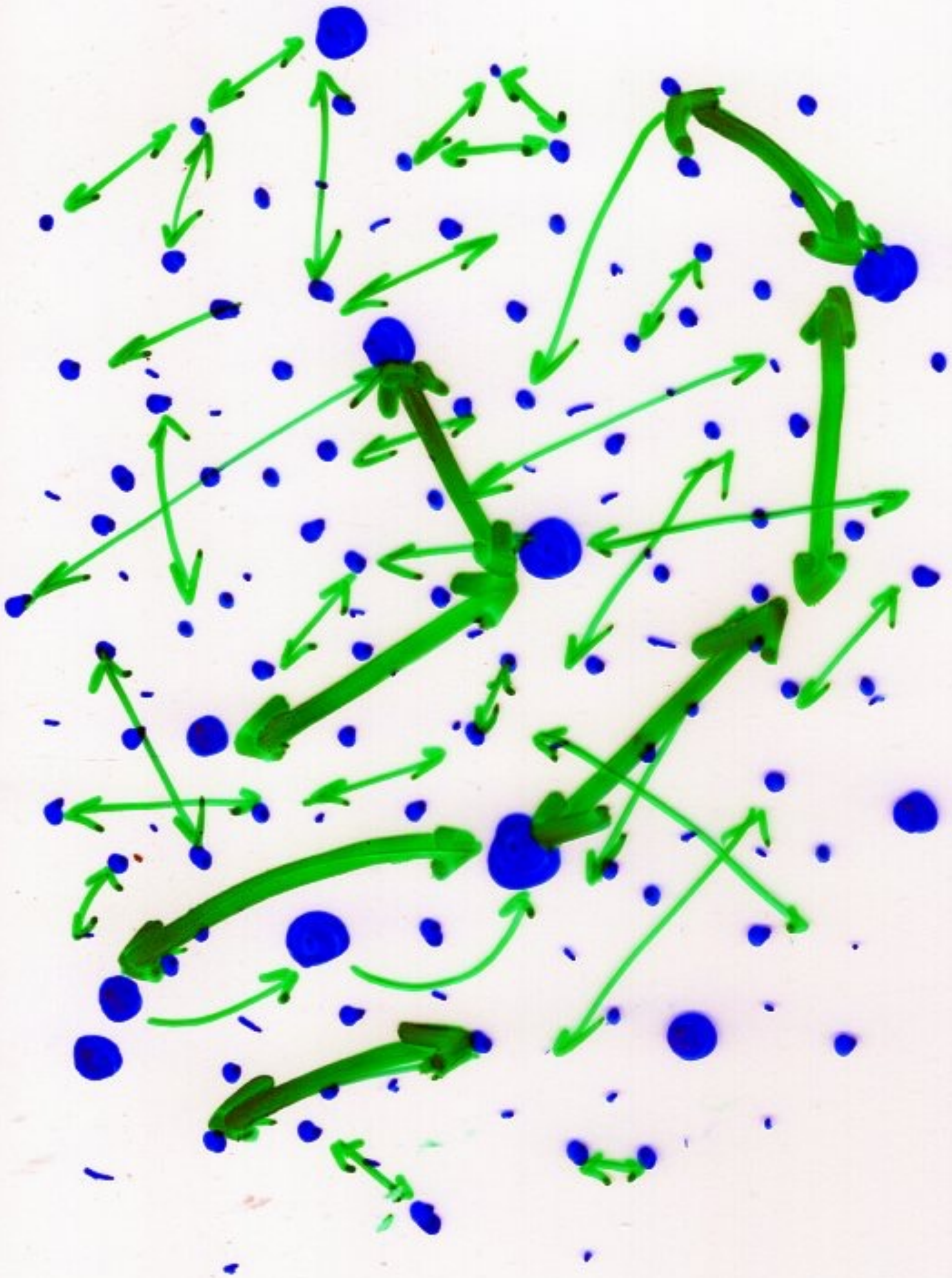


many formulae, identities ...



many
formulae,
identities ...

many
bijections
correspondences ...



many
formulae,
identities ...

many
bijections
correspondences ...
basic bijections

many
formulae,
identities ...

many
bijections
correspondences ...
basic bijections
"bijective tools"
or "basic Lemma"

example:

hook-lengths formula

and

tilings of the Aztec diagram

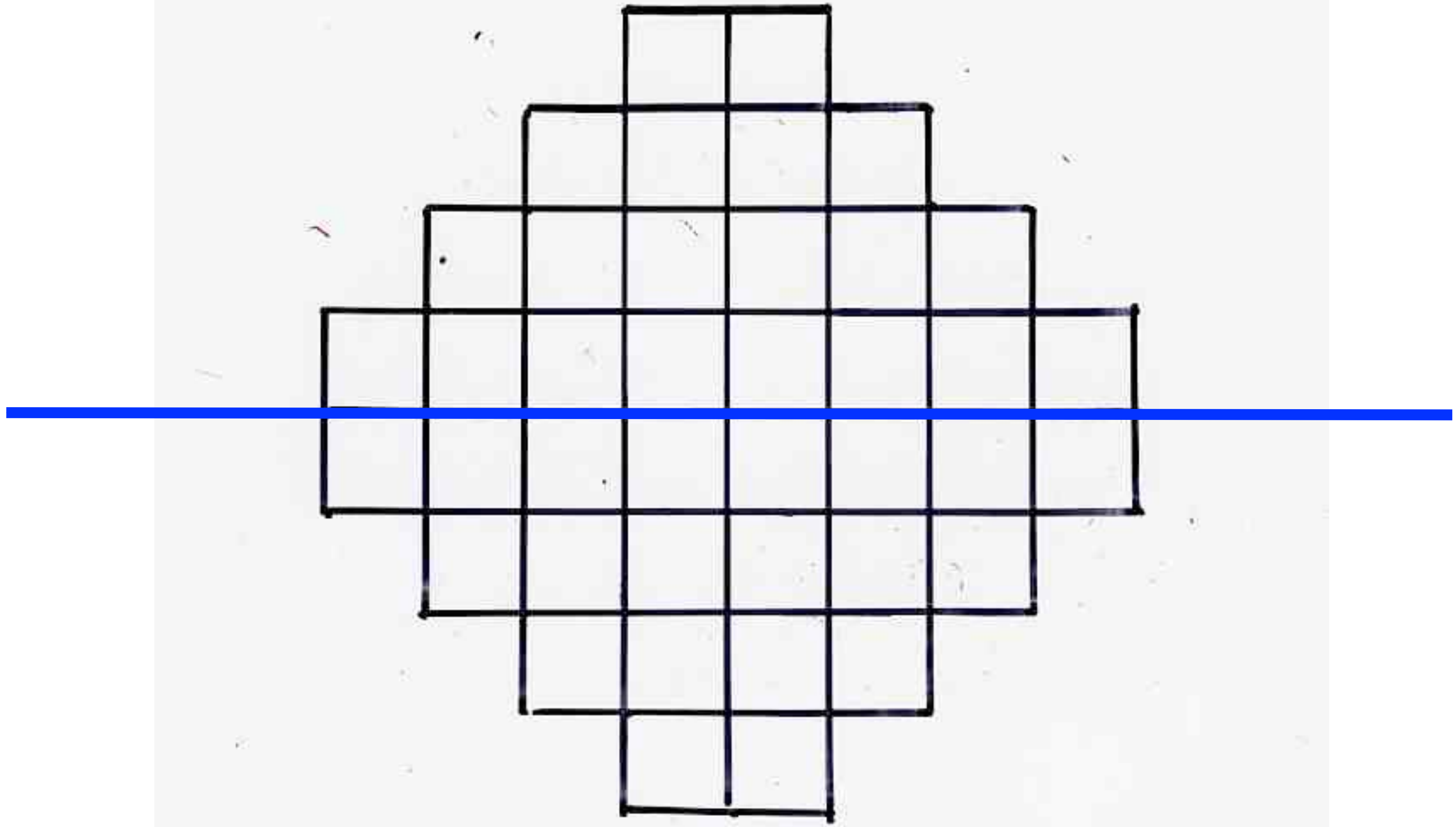
under the same roof

2	1			
3	2			
5	4	1		
8	7	4	2	1

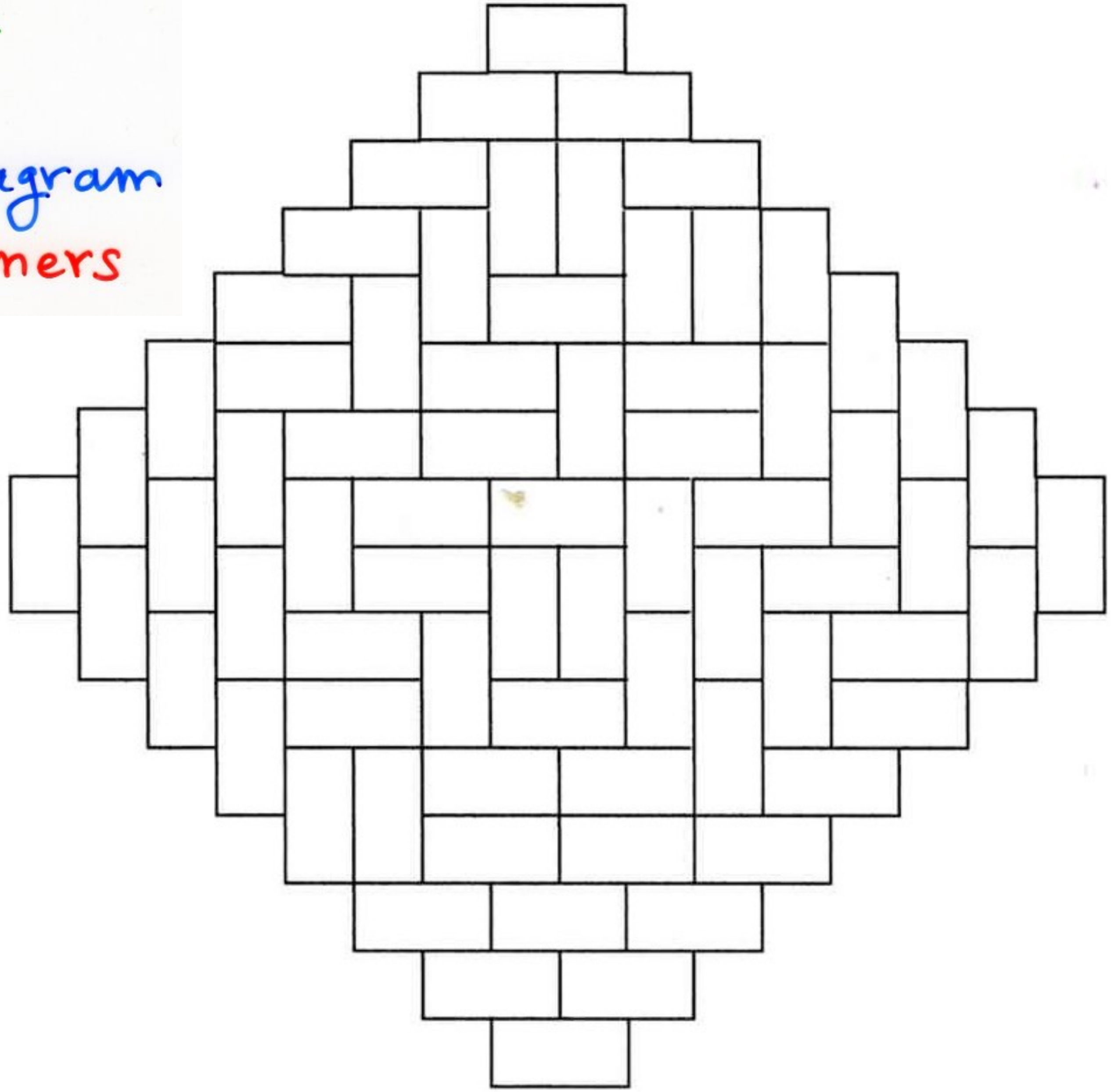
$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula

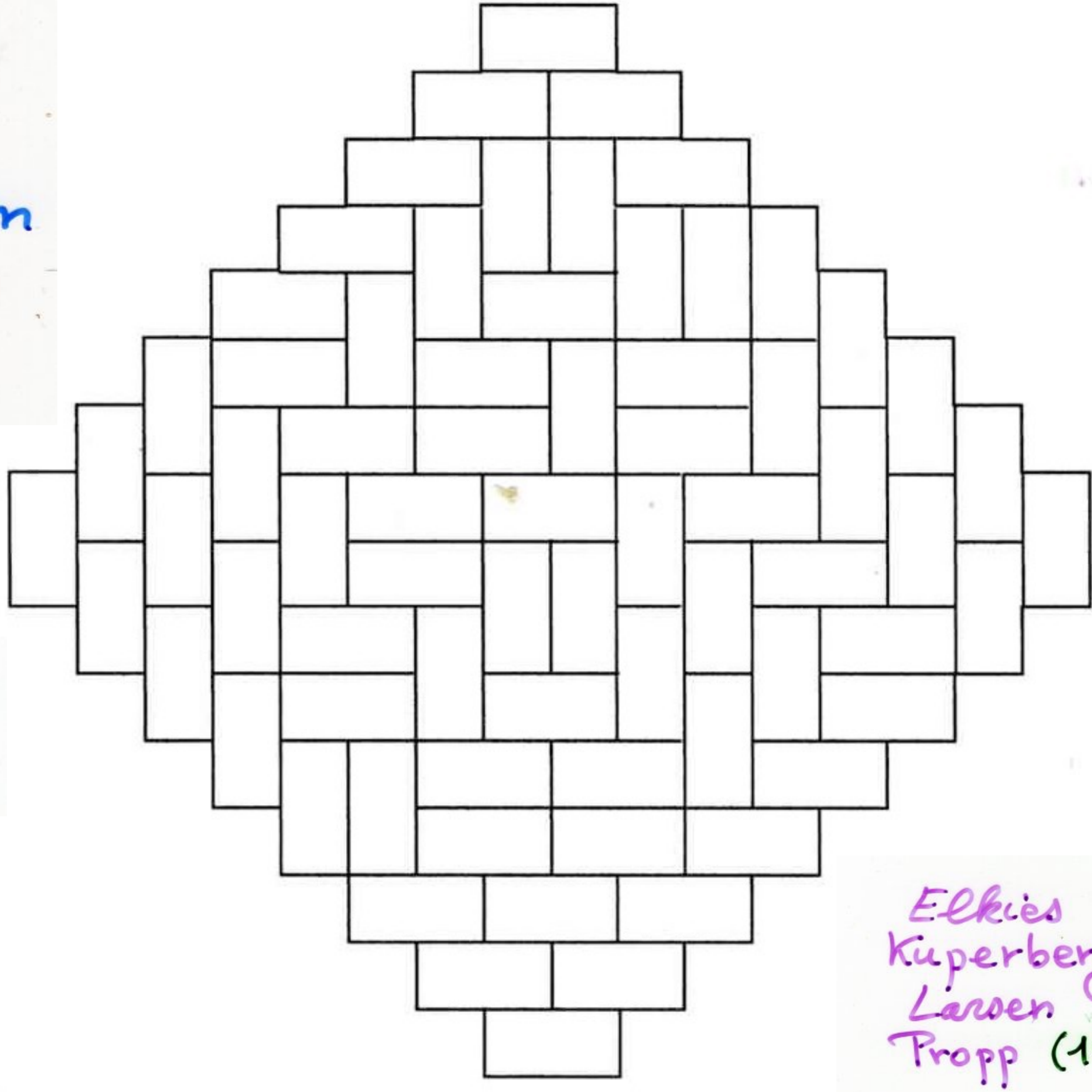
Aztec diagram



tiling
of the
Aztec diagram
with dimers



the number
of tilings
of the
Aztec diagram
with dimers
is

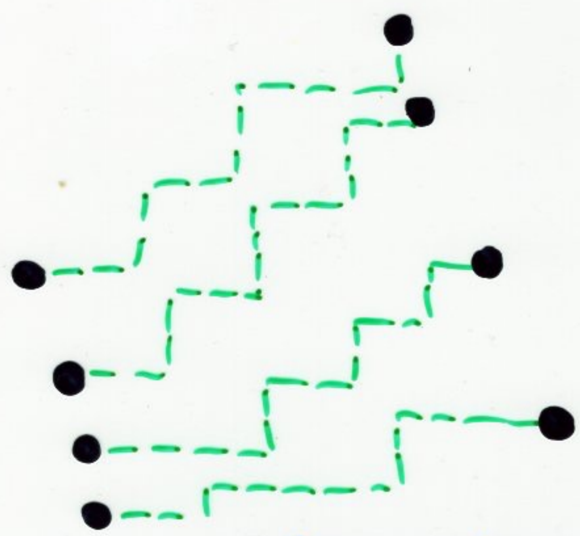


$$2^{(1+2+3+\dots+n)}$$

$$2^{\frac{n(n+1)}{2}}$$

Elkies
Kuperberg
Larsen
Propp (1992)

LGV- Lemma

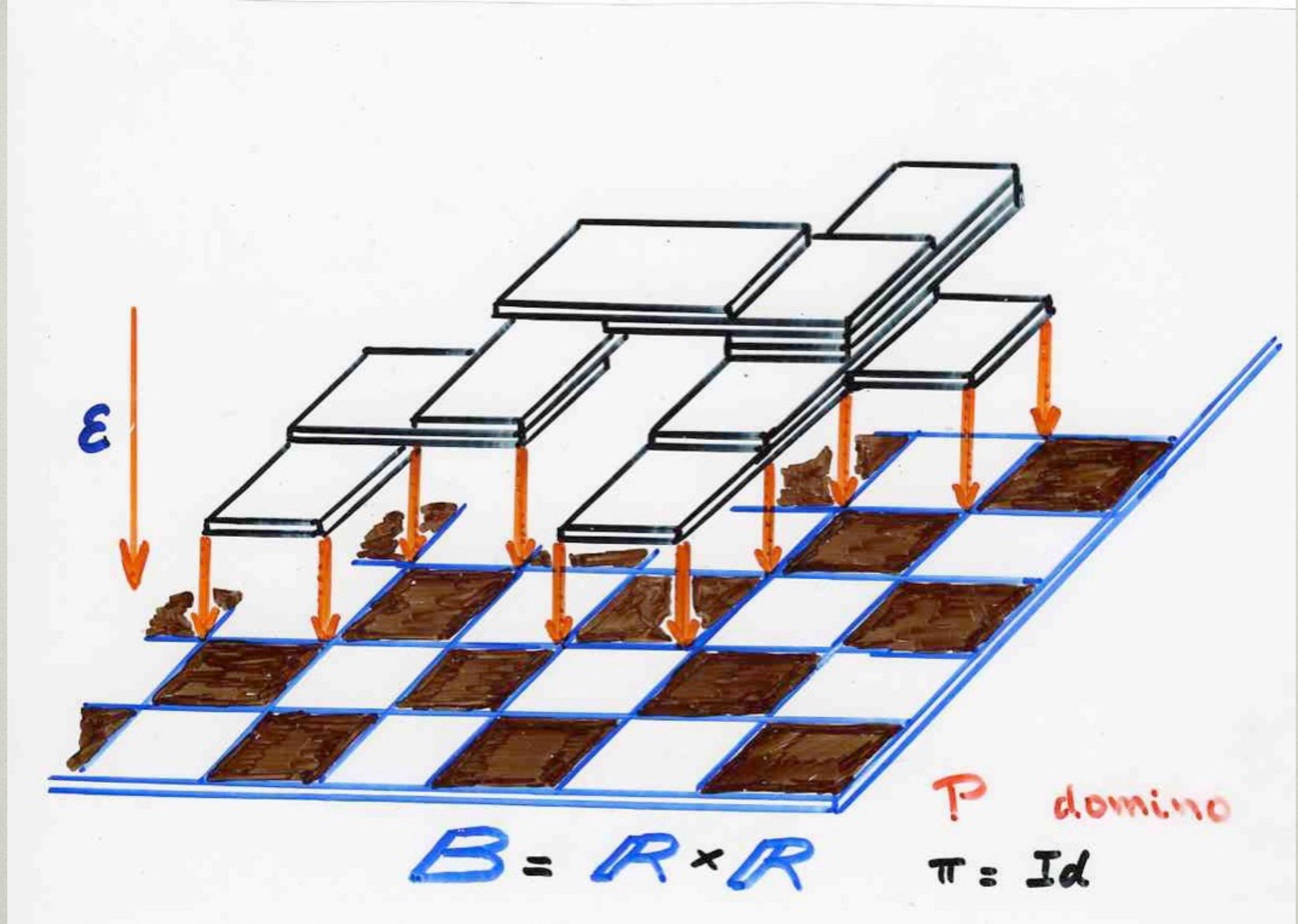
determinant = 

non-intersecting **paths**

(under certain conditions)

another example:

Heaps of pieces



ε
P domino
$B = \mathbb{R} \times \mathbb{R}$
$\pi = Id$

the course:

ch1. ordinary generating function

ch2. exponential generating function

ch3. Bijections for the Catalan garden

ch4. Bijections for the n! garden

ch5. Tilings, determinant and non-intersecting paths

ch6. (?) Combinatorial theory of differential equations other courses:

(2017) Heaps of pieces and interactions
(physics, algebra, ...)

(2018) Combinatorial theory of orthogonal polynomials
and continued fractions

(2019) The « cellular ansatz »:
quadratic algebra and tableaux (combinatorial objects
drawn on a planar lattice), applications to physics