Course IMSc Chennai, India
January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:
commutations and heaps of pieces
(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

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Chapter 6

Heaps and Coxeter groups

(2)

fully commutative elements
and Temperley-Lieb algebra

IMSc, Chennai

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from the previous lecture
Symmetric group $\mathcal{S}_n$

$n!$ permutations

$\sigma_i = (i, i+1)$ for $i=1, 2, \ldots, n-1$

transposition of two consecutive elements

\[
\begin{cases}
(i) & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2 \\
(ii) & \sigma_i^2 = 1 \\
(iii) & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.
\end{cases}
\]

Moore-Coxeter
Yang-Baxter

Coxeter graph

\[1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8\]
heaps of dimers \([1,n]\) on \(\{0,1,\ldots,n-1\}\) generators \(\tau_0, \tau_1, \ldots, \tau_{n-1}\) with \(\tau_i \tau_j = \tau_j \tau_i \) if \(|i-j| \geq 2\)

heap of dimers \(\rightarrow\) permutation \(S_n\)
heap of dimers $[1,n] \rightarrow$ permutation $S_n$
reduced decomposition of a permutation
\[ \sigma = u_{i_1} \ldots u_{i_k} \]

\( k \) minimum (nb of inversion)

Lemma. The set \( R(w) \) of reduced decompositions is a disjoint union of commutation classes.

For each of them, there exist a heap \( H(C) \) of \( H(w,s) \) such that \( C \) is exactly the set of linear extensions of the poset \( H(C) \)
Definition. An element $w$ of the Coxeter group $\mathcal{W}$ is fully commutative iff $R(w)$ is reduced to one commutation class.

The corresponding heap $H(w)$ will also be called fully commutative (FC).
the stairs decomposition of a heap of dimers
Catalan numbers

\[ 1 \leq \frac{2}{1} \leq \frac{3}{2} \leq \frac{4}{3} \leq \frac{7}{4} \leq \frac{12}{7} \leq \frac{13}{12} \leq \frac{15}{13} \leq \frac{n}{15} \leq n \]
bijection

fully commutative heaps

321-avoiding permutations
(321) - avoiding permutations

no occurrences of

\[ \sigma(i) > \sigma(j) > \sigma(k) \]

\[ i < j < k \]
Prop \( \sigma \in S_n \) permutation

- (321) - avoiding
- only one commutation class

(Billey, Jockusch, Stanley) (1993)

\[
\text{counted by } \quad C_n = \frac{1}{n+1} \binom{2n}{n}
\]

Catalan numbers
Temperley-Lieb algebra
Temperley–Lieb algebra

\( TL_n(\beta) \)

generators \( \{e_1, e_2, \ldots, e_{n-1}\} \)

(i) \( e_i e_j = e_j e_i \quad |i-j| \geq 2 \)

(ii) \( e_i^2 = \beta e_i \)

(iii) \[
\begin{cases}
    e_i e_{i+1} e_i = e_i & \\
    e_{i+1} e_i e_{i+1} = e_{i+1}
\end{cases}
\]

\( \beta \) scalar
\[
(1 + 3 e_2 e_3 e_1 + 2 \underbrace{e_2 e_3 e_2 e_3}_{e_2}) \times \\
(1 - e_2 + 4 \underbrace{e_3 e_1 e_3}_{e_3 e_1})
\]
sequence of rewrites (reductions)

\[ w \xrightarrow{\ast} \bar{w} \]

and commutations

\[ e_i \xrightarrow{\beta} \tilde{e}_i \]

\[ e_i, e_{i+1}, e_i \xrightarrow{} e_i \]

\[ e_{i+1} e_i, e_{i+1} \xrightarrow{} e_{i+1} \]
\[ w = e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_4 e_2 \]

\[ \beta e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_4 e_2 \]

\[ \beta e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ e_1 e_2 e_4 e_1 e_3 e_2 e_4 e_2 \]

\[ \bar{w} = \beta^2 e_1 e_4 e_2 \]
Definition: A word $w$ is reduced off each word of the commutation class $[w]$ has no factors of the type: $e_i e_i, e_i e_i e_i, e_i e_i e_i e_i$ (no possible rewriting).

$C$ commutations $e_i e_j = e_j e_i$ with $|i-j| \geq 2$.

Proposition: If $w_1, w_2$ reduced, $i = j$ and $w_1 \equiv_C w_2$, then $w_1 \xrightarrow{\beta^j w_1} \xrightarrow{\beta^i w_2}$, with $\beta^j w_1, \beta^i w_2$.
Definition \( H \) reduced heap of dimers on \( \mathbb{N} \) iff no factor \( H = H'FH'' \) with

\[
F = A \ X \ H \ X \ X
\]

\( i \leq i' \leq i+1 \) and \( i+1 \leq i'' \leq i+2 \).
Proposition

If \( H \) heap \( \xrightarrow{\beta_i} H_1 \) and \( \xrightarrow{\beta_j} H_2 \) with reduced heaps then \( H_1 = H_2 \)

Heap \( H \) \( \xrightarrow{D} \) Planar diagram \( D(H) \)
heap $H$ $\xrightarrow{D}$ planar diagram $D(H)$
Proposition. If $H \xrightarrow{\text{heap}} H_1 \xleftarrow{\beta^i} H_2$ with reduced heaps then $H_1 = H_2$.

Proposition. The restriction of the map $D$ to reduced heaps is a bijection $D$ reduced heap $\leftrightarrow$ planar diagrams (no loops).
exercise: give a proof of the last proposition
Proposition If \( \beta^i w_1 \) and \( \beta^j w_2 \) are reduced, then \( i = j \) and \( w_1 \equiv_w w_2 \).

\text{C commutations } e_i e_j = e_j e_i \text{ with } |i-j| \geq 2.
enumerated by Catalan numbers
in $\text{TL}_n(\beta)$
Temperley-Lieb
algebra

product
of two elements
product
of two elements

in $TL_n(p^2)$
Temperley-Lieb
algebra
$e_i^2 = \beta e_i$

$e_i e_{i+1} e_i = e_i$

Kaufman generators

$|i-j| \geq 2$
Basis of Temperley-Lieb algebra
basis of \( (N)TL_n \)

no occurrence of \( L \)

\( u_i^2 \) \( u_i u_{i+1}^2 \) \( u_{i+1}^2 \) \( u_{i+1} u_i \) \( u_{i+1} u_i u_{i+1} \)

strict heap
$1 \leq \frac{2}{x} < \frac{3}{x} < \frac{4}{x} < \frac{7}{x} < \frac{12}{x} < \frac{13}{x} < \frac{15}{x} \leq n$
\((e_2 e_4) (e_3 e_2) (e_4) (e_7) (e_{12} e_{11} e_{10} e_7 e_8) \times \\
\times (e_{13} e_{12} e_{11}) (e_{15})\)
$1 \leq n \leq 2 \leq 3 \leq 4 \leq 7 \leq 12 \leq 13 \leq 15$
heap of dimers on $[0, n-1]$ $\rightarrow$ element of $T_{2n}$ Temperley-Lieb algebra
exercise:

RSK

and fully commutative heaps
Robinson–Schensted
Catalan numbers
nil-Temperley-Lieb algebra
nil-Temperley-Lieb algebra

$\text{NTL}_n$ or $A_n^0$

(i) $e_i e_j = e_j e_i$ for $|i-j| > 2$

(ii) $e_i^2 = 0$

(iii) $e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = 0$
**nil-Temperley-Lieb algebra**

\[ NTL_n \text{ or } A_n^0 \]

\[(i)\quad e_i e_j = e_j e_i \quad (|i-j| \geq 2) \]

\[(ii)\quad e_i^2 = 0 \]

\[(iii)\quad e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = 0 \]

**same dimension**

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]
basis of $N(TL_n)$

no occurrence of

strict heap

$u_i$, $u_{i+1}$, $u_i$

$u_{i+1}$, $u_i$, $u_{i+1}$
representation of $\mathbf{NTL}_n$

with operators on the Young lattice
v_i(λ) = \begin{cases} 1 & \text{if } \lambda = \lambda_i \\ 0 & \text{else} \end{cases}
(i) \( v_i \) \( v_j \)

(ii) \( v_i^2 \)

(iii) \( v_i, v_t, v_i \)

\( v_t, v_i, v_{i+1} \)
next lecture

Chapter 7

Heaps in physics