Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 1
Paths and moments

Ch 1a

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orthogonal polynomials: 4 examples

Tchebychev polynomials — 1st kind, 2nd kind

Hermite polynomials

Laguerre polynomials

Tchebychev polynomials

1st kind

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

Tchebychev

Pafnouti Lvotich

Chebyshev

П. Л. ЧЕБЫШЁВ

$$\sin\big((n+1)\theta\big) = \sin\theta \; U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind

sequence of orthogonal polynomials

$$\int_{-1}^{+1} U_m(x)\, U_n(x)\, (1-x^2)^{1/2} = \frac{\pi}{2}\, \delta_{m,n}$$

$$U_n(x) = S_n(2x)$$

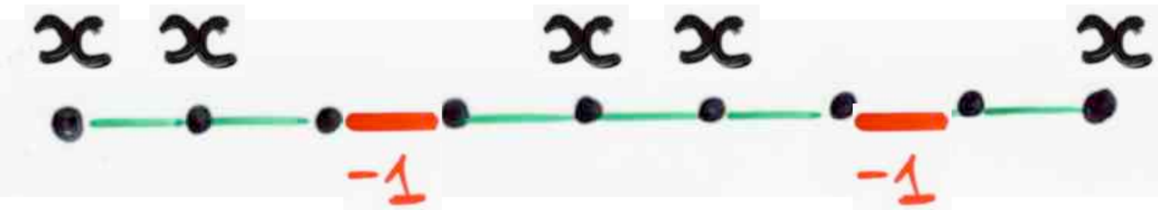
$S_n(x)$ matching polynomial of the segment graph Seg_n

$a_{n,k}$ = number of matchings of $[0, n-1]$ with k dimers

$$S_n(x) = \sum_{k \geqslant 0} (-1)^k a_{n,k} \, x^{n-2k}$$

$$S_n(x) = \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0,\, n-1]}} (-1)^{|\alpha|}\, x^{n-2|\alpha|}$$

$|\alpha|$ = number of dimers of α



$$= \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0,\, n-1]}} (-1)^{|\alpha|}\, x^{ip(\alpha)}$$

$ip(\alpha)$ = number of isolated points of α

$= n - 2|\alpha|$

Fibonacci
and
Tchebychev polynomials

$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k$$

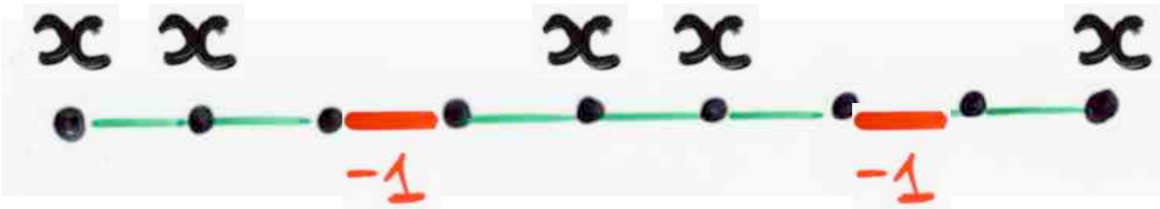
$$= \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0,\, n-1]}} (-x)^{|\alpha|}$$

$$F_{n+1}(x) = F_n(x) - x F_{n-1}(x)$$

$$F_0 = F_1 = 1$$

$$F_{n+1} = F_n + F_{n-1}$$

$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k \qquad = \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0, n-1]}} (-x)^{|\alpha|}$$



$S_n(x)$

$$S_n^*(x) = x^n S_n(1/x) \qquad = \sum_{\substack{\alpha \\ \text{matching} \\ \text{of } [0, .., n-1]}} (-x^2)^{|\alpha|} \qquad = F_n(x^2)$$

reciprocal
polynomial

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

$$U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x)$$

$U_0(x) = 1$, $U_1(x) = 2x$

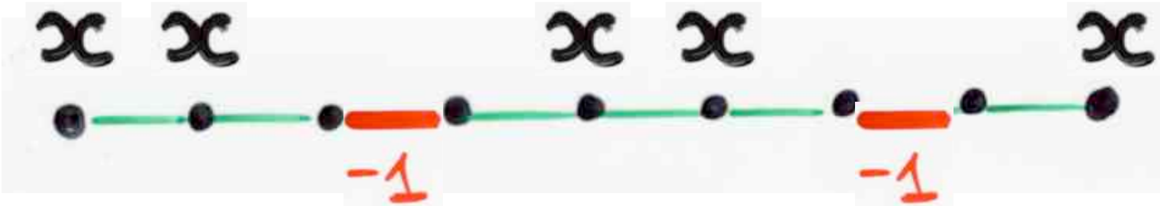
same 3-terms recurrence relation

$$U_n(x) = S_n(2x)$$

$$U_n(x) = S_n(2x)$$

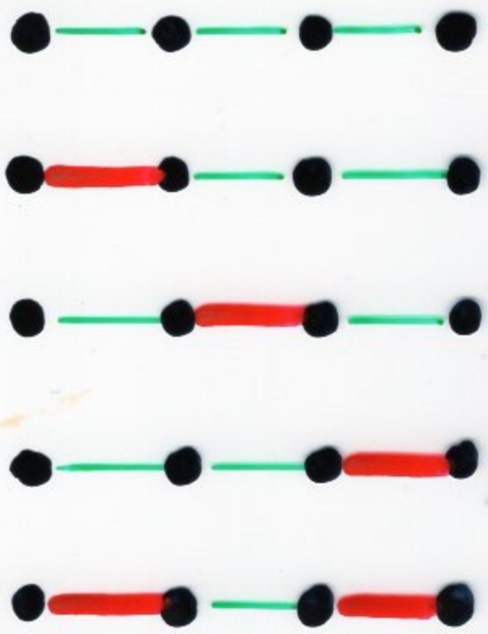
$S_n(x)$

matching polynomial of the segment graph Seg_n



$$S_{n+1}(x) = x\,S_n(x) - S_{n-1}(x)$$

$$S_0(x) = 1, \quad S_1(x) = x$$



$$x^4 \qquad (2\cos\theta)^4$$

$$-3x^2 \qquad -3(2\cos\theta)^2$$

$$+1 \qquad 1$$

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$$

exercise

Prove (bijectively!)

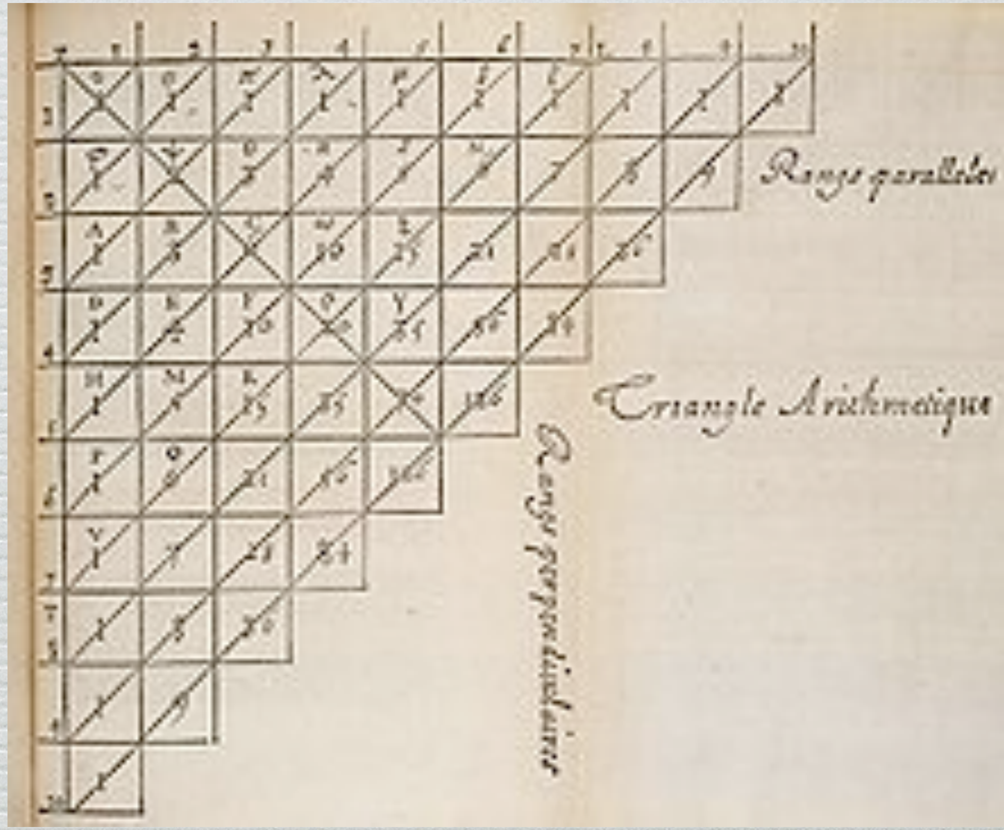
$$a_{n,k} = \binom{n-k}{k}$$

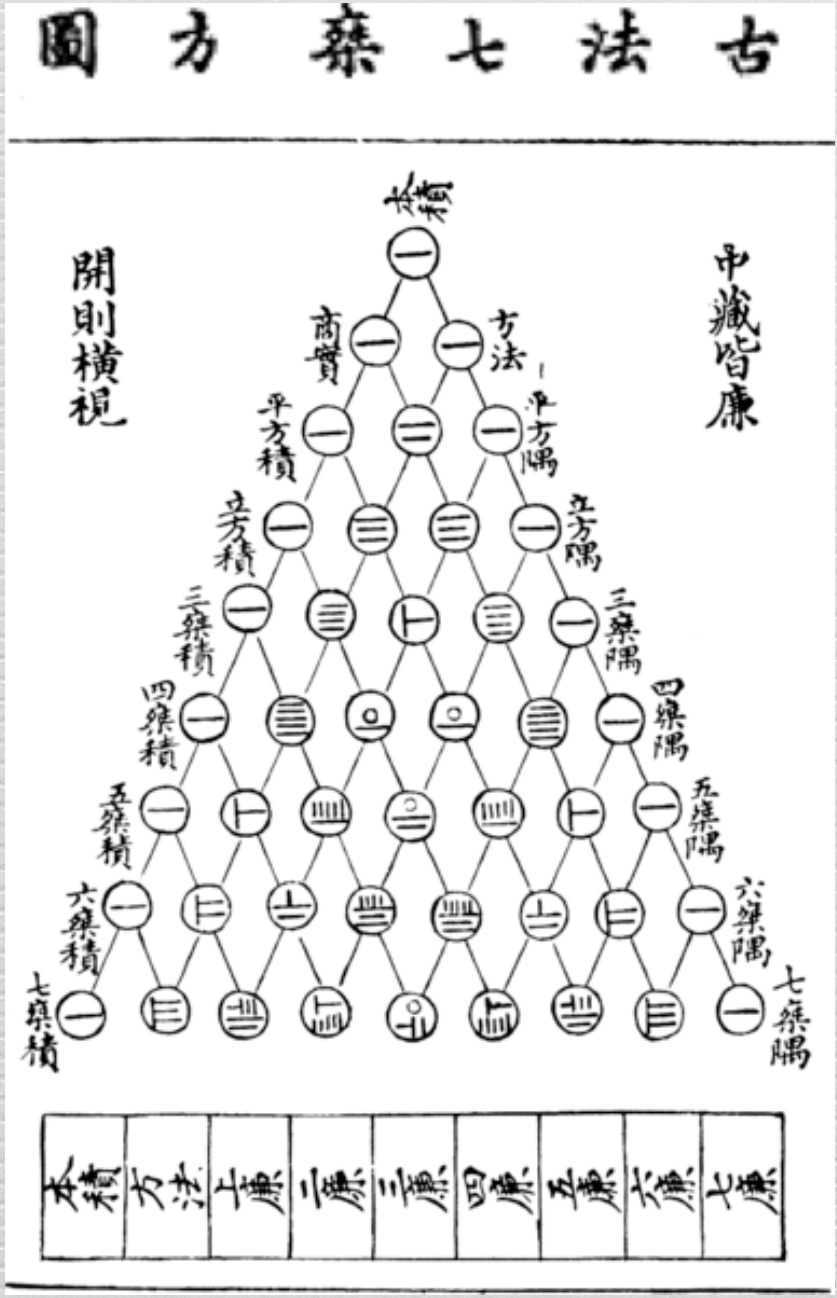
$$S_n(x) = \sum_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$$

addition +

	1								
1	1	1							
1	1	2	1						
2	1	3	3	1					
3	1	4	6	4	1				
5	1	5	10	10	5	1			
8	1	6	15	20	15	6	1		
13	1	7	21	35	35	21	7	1	
21	1	8	28	56	70	56	28	8	1

Pascal triangle
binomial coefficients





Yang Hui triangle
(11th, 12th century)

in Persia
Omar Khayyam
(1048-1131)

in India
Chandas Shastra by Pingala
2nd century BC

Pingala (2nd century B

Pingala   Laghu (short syllabe)
            Guru (long syllabe)

two classes of meters in Sanskrit

- Akṣarachandaḥ
  Chandaḥ number of syllables
  later 4 feet (pāda)
- number of mātrās (time measure)
  short syllabe : one mātrās
  long syllabe : two mātrās

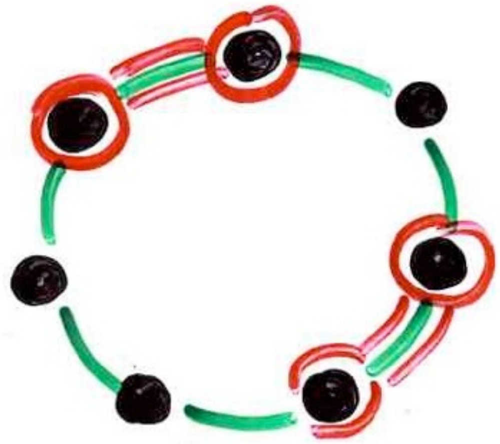
relation with Fibonacci numbers ?

Examples:
some orthogonal polynomials

Tchebychev polynomials

2nd kind

$$\cos(n\theta) = T_n(\cos\theta)$$



$T_n(x)$

Tchebychef

polynomial

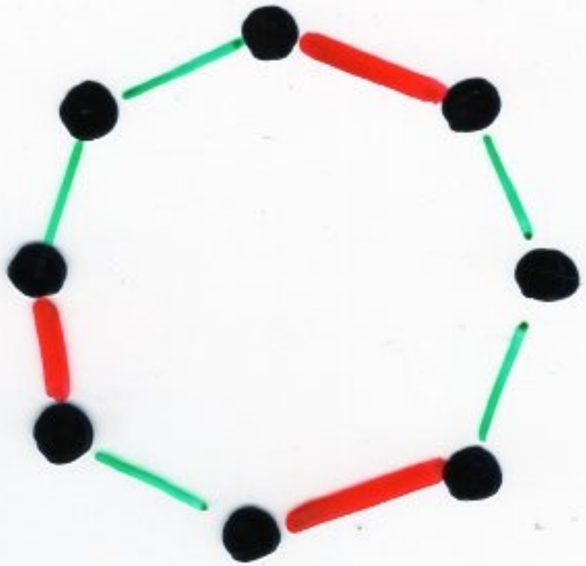
1st kind

$$\int_{-1}^{+1} T_m(x)\, T_n(x)\, (1-x^2)^{-1/2} = \begin{cases} \frac{\pi}{2}\, \delta_{m,n}, & n \neq 0 \\ \pi\, \delta_{m,n}, & n = 0 \end{cases}$$

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$

$$T_0(x) = 1, \quad T_1(x) = x$$

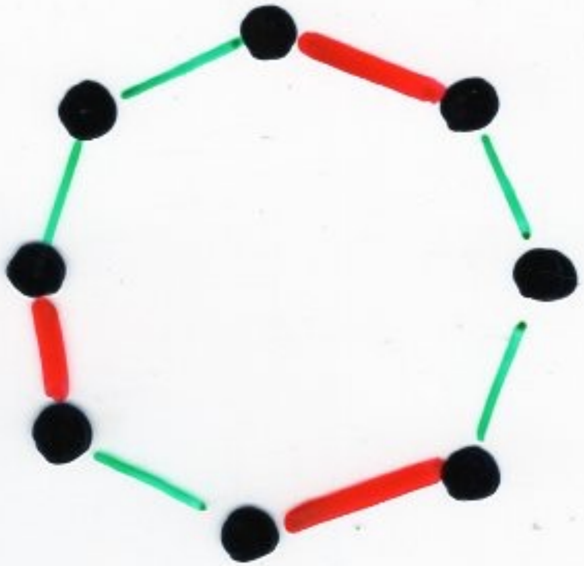
Lucas
and
Tchebychev polynomials



matching polynomial
of the cycle graph

$$C_n(x) = \sum_{\substack{\text{matching } M \\ \text{of } \gamma}} (-1)^{|M|} \, x^{\,ip(M)}$$

number of isolated points



matching polynomial of the cycle graph

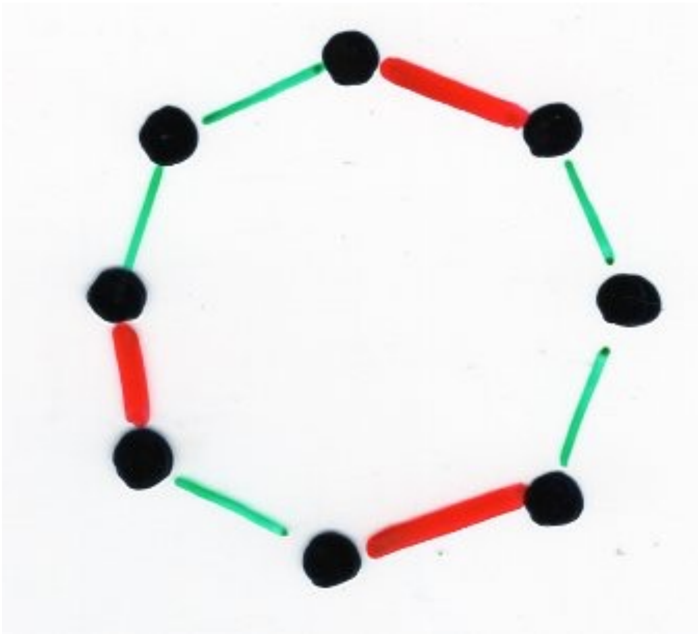
$$T_n(x) = \frac{1}{2} C_n(2x)$$

$$C_{n+1}(x) = x\,C_n(x) - \lambda_n C_{n-1}(x)$$

same 3-terms recurrence relation

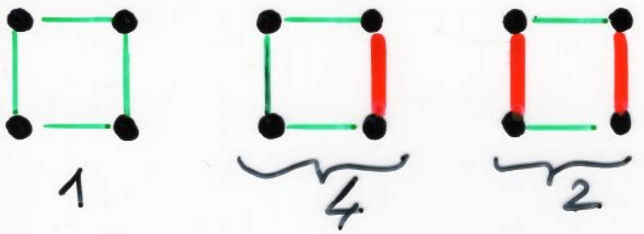
$$\begin{cases} C_0 = 1 \\ C_1 = x \end{cases}$$

$$\begin{cases} \lambda_1 = 2 \\ \lambda_n = 1 \quad (n \geq 2) \end{cases}$$



Lucas polynomial

$$L_n(x) = \sum_{\substack{\text{matchings } M \\ \text{of a cycle } \gamma \\ \text{length } n}} (-x)^{|M|}$$

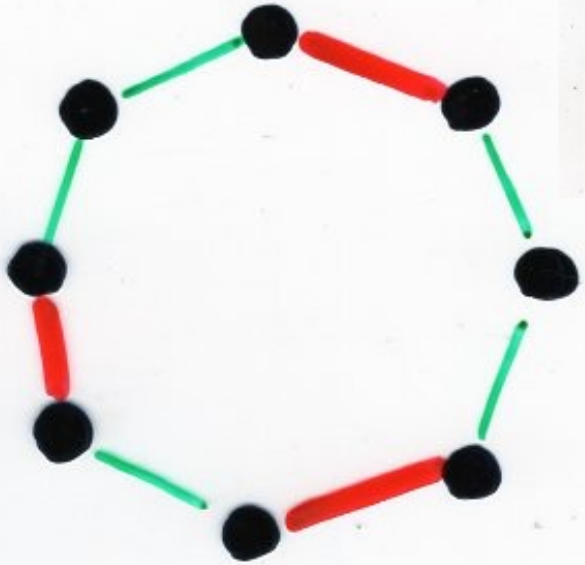


$$L_4(x) = 1 \quad -4x \quad +2x^2$$

reciprocal of $L_n(x^2)$ is

$$C_n(x) = \sum_{\substack{\text{matching } M \\ \text{of } \gamma}} (-1)^{|M|}\, x^{ip(M)}$$

number of isolated points



G = "cycle graph" Γ_n

matching polynomial of the cycle graph Γ_n

$$C_n(x) = M_{\Gamma_n}(x)$$

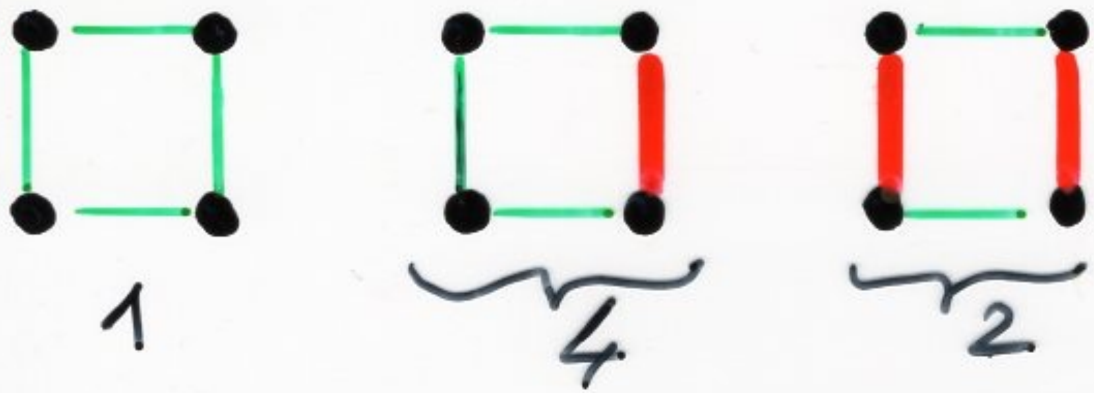
$$\cos(n\theta) = T_n(\cos\theta)$$

$$C^*_n(x) = L_n(x^2)$$

reciprocal polynomial

Lucas polynomial

$$T_n(x) = \frac{1}{2} C_n(2x)$$



$$L_4(x) = 1 \quad - 4x \quad + 2x^2$$

$$L_4(x^2) = 1 \quad - 4x^2 \quad + 2x^4$$

$$L_4^*(x^2) = x^4 \quad - 4x^2 \quad + 2$$

$$C_4(2) = 16x^4 \quad - 16x^2 \quad + 2$$

$$\left(8\cos^4\theta \quad - 8\cos^2\theta \quad + 1\right) = \cos 4\theta$$

$$T_4(\cos\theta)$$

exercise

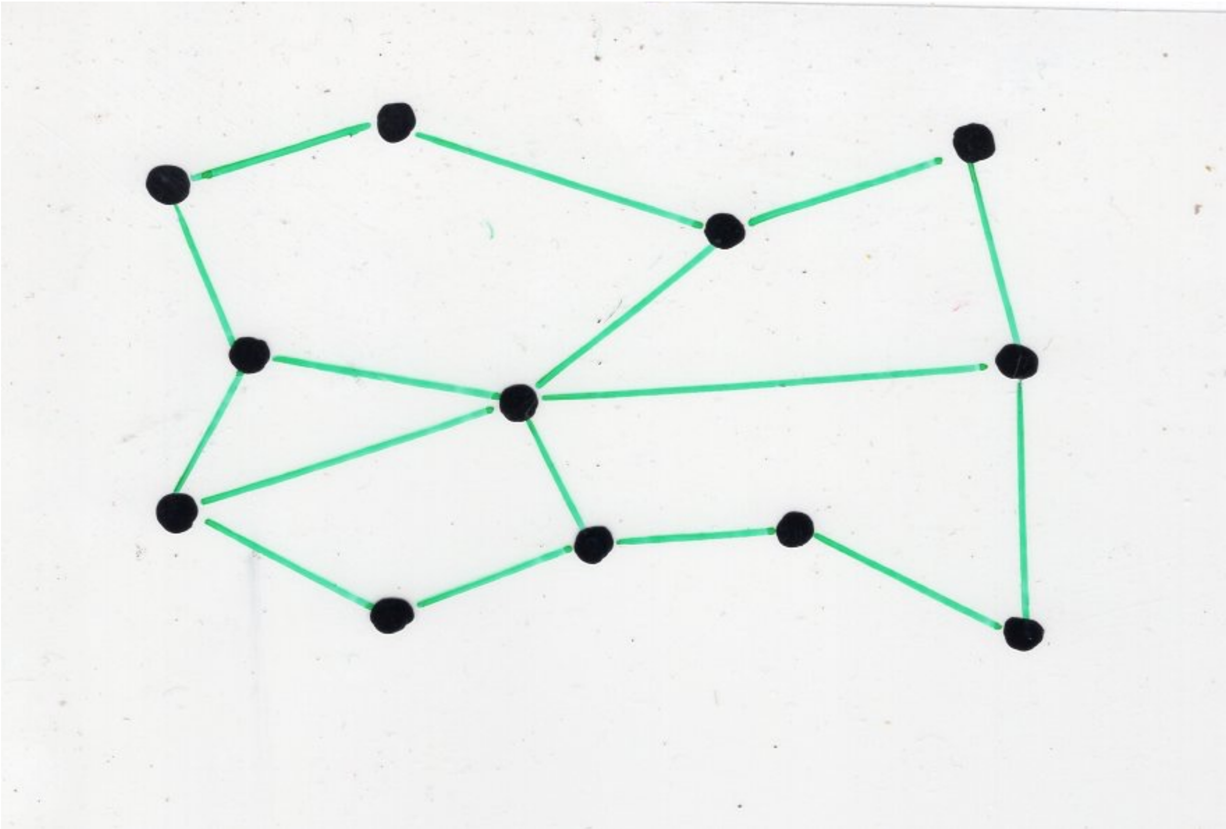
prove (bijectively!)

$$\frac{n}{n-k}\binom{n-k}{k} = \text{number of matching of } \Gamma_n \text{ having } k \text{ dimers}$$

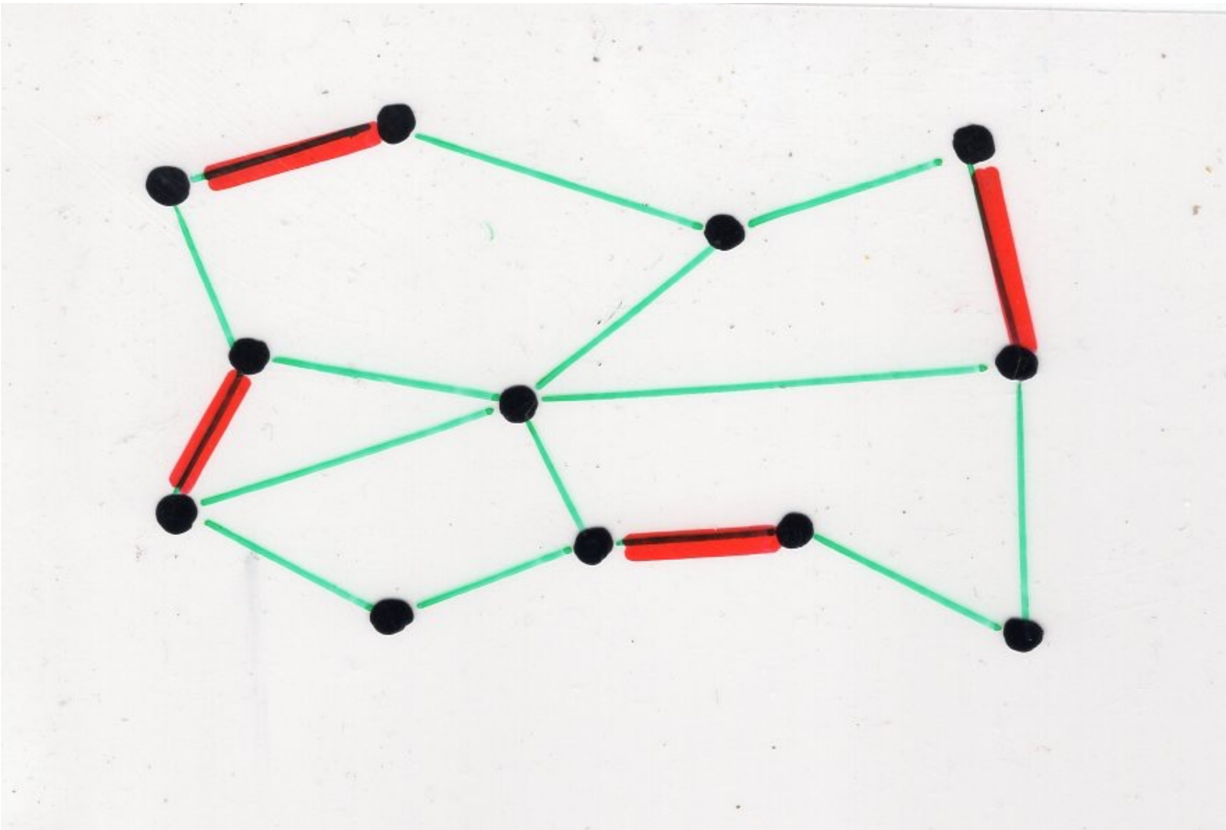
Γ_n cycle graph
n vertices

$$C_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}$$

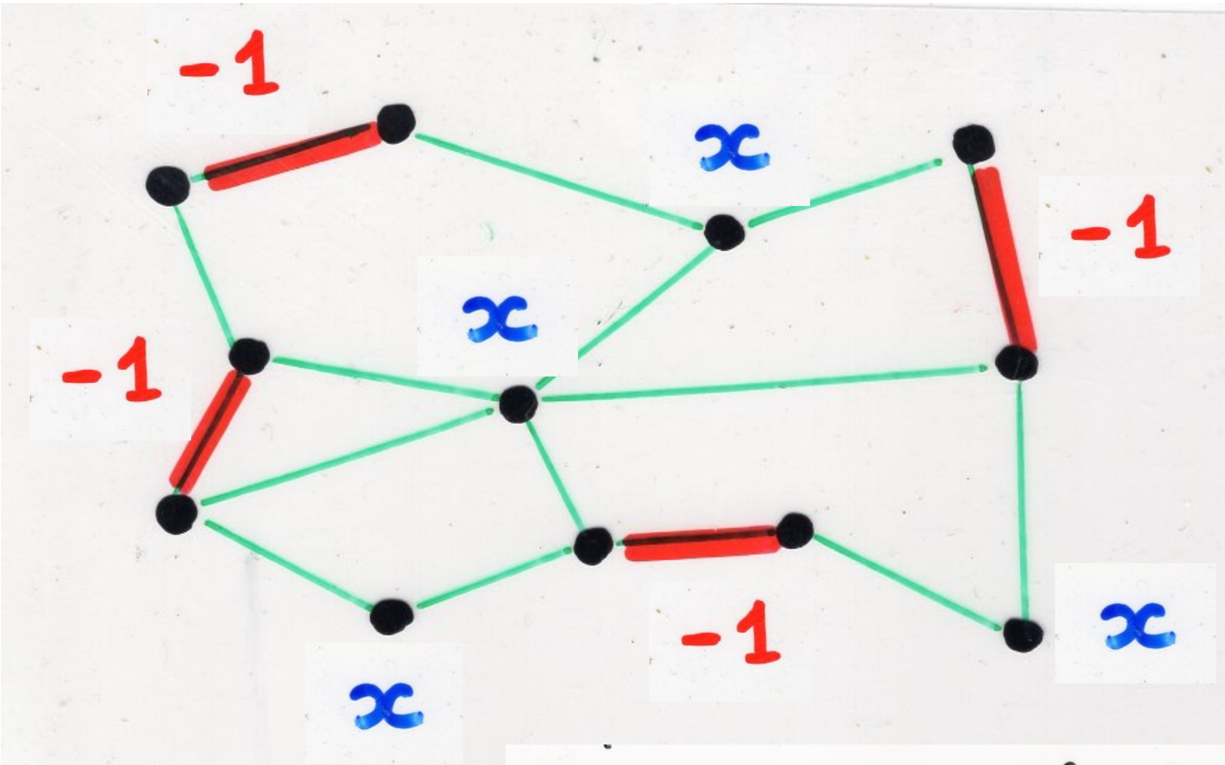
Matching polynomial of a graph



matching
polynomial
of a graph G



matching of a graph G = set of 2 by 2 disjoint edges



Matching polynomial of a graph G

$$M_G(x) = \sum_{\substack{\text{matchings } M \\ \text{of } G}} (-1)^{|M|} x^{ip(M)}$$

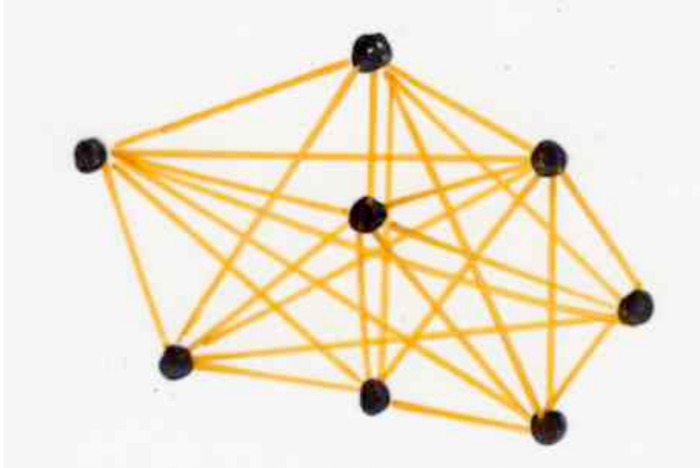
$ip(M) =$ number of isolated vertices of G

$$= \sum_{M} (-1)^{|M|} x^{n-2|M|}$$

$n =$ nb of vertices of G

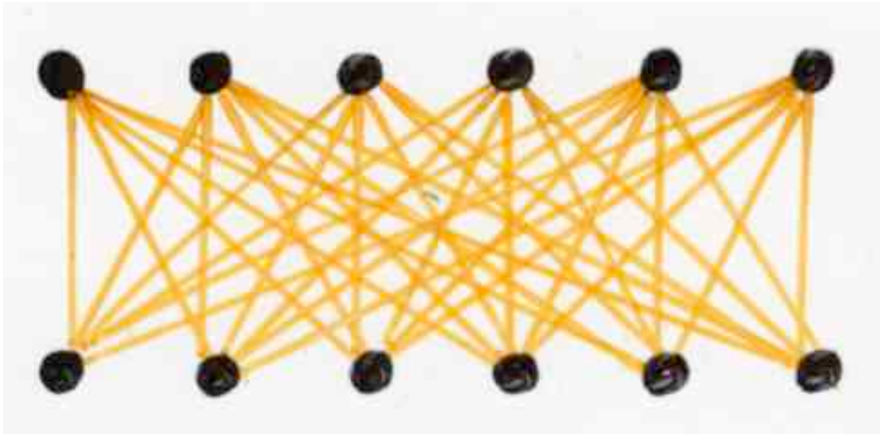
Hermite polynomial $H_n(x)$

matching polynomial of the complete graph K_n



Laguerre polynomial $L_n(x)$ = matching polynomial of $K_{n,n}$

complete bipartite graph



Hermite polynomials

Charles Hermite
1822 - 1901

$$\mathbf{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

Rodrigues

$$\int_{-\infty}^{+\infty} \mathbf{H}_n(x)\,\mathbf{H}_m(x)\, e^{-x^2/2}\, dx = \sqrt{\pi}\; n!\; \delta_{n,m}$$

"physicists" Hermite polynomial $H_n(x)$

$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp(2xt - t^2)$$

"probabilists' Hermite polynomial" $He_n(x)$

(combinatorial) Hermite polynomials

$$\sum_{n\geq 0} He_n(x)\frac{t^n}{n!} = e^{\left(xt - \frac{t^2}{2}\right)}$$

"physicists" Hermite polynomial $H_n(x)$

"probabilists' Hermite" polynomial $He_n(x)$

(combinatorial) Hermite polynomials

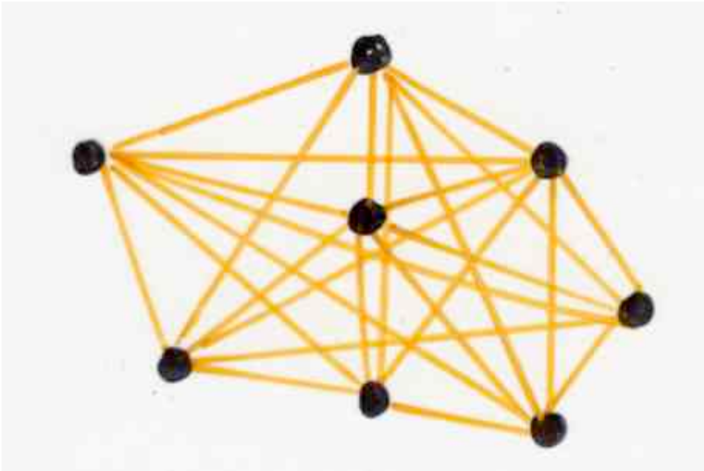
$$H_n(x) = 2^{n/2}\, He_n(\sqrt{2}\, x)$$
$$He_n(x) = 2^{n/2}\, H_n(x/\sqrt{2})$$

<u>notation</u>

$H_n(x)$ means $He_n(x)$ in this course (as in Part I, II, III)

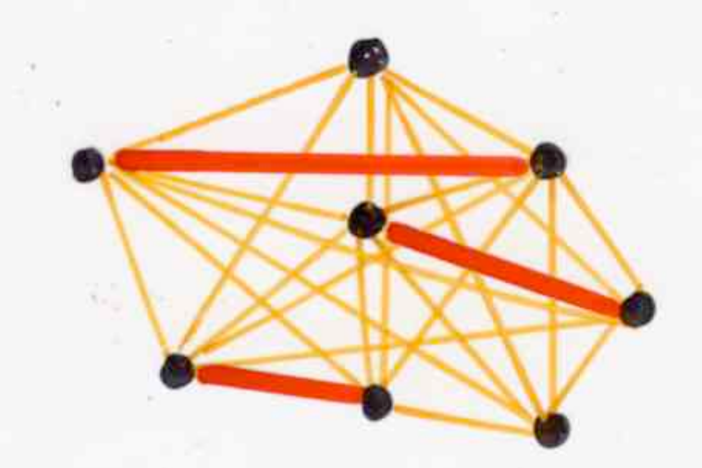
Hermite polynomial $H_n(x)$

matching polynomial
of the complete graph K_n



Hermite polynomial

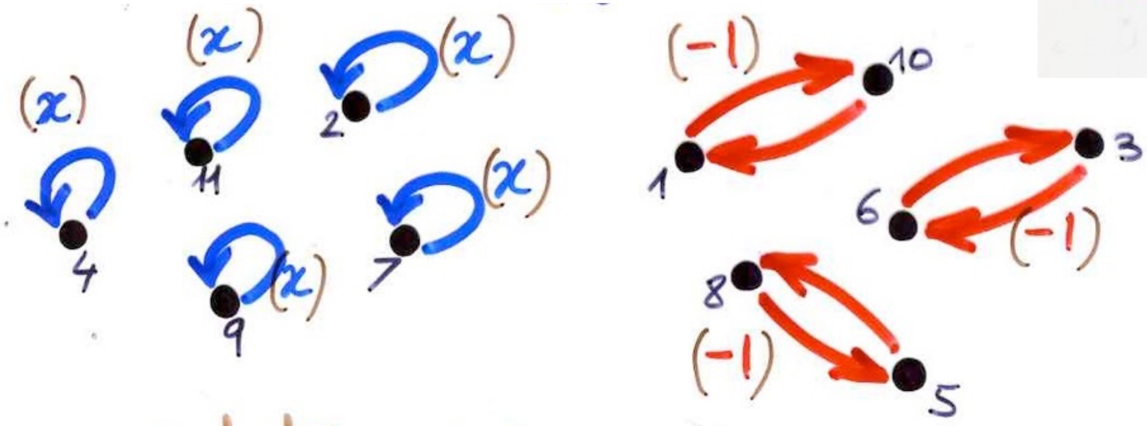
matching polynomial of the complete graph K_n



$ip(\alpha) = n - 2|\alpha|$

$$H_n(x) = \sum_{\substack{\alpha \\ \text{matching of } K_n}} (-1)^{|\alpha|} x^{ip(\alpha)}$$

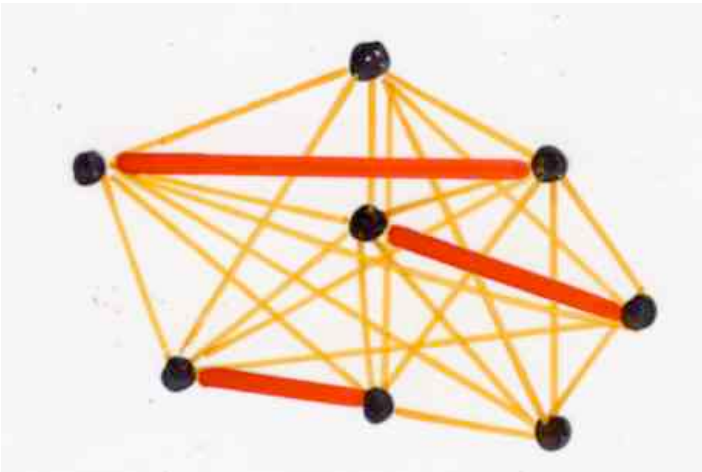
Hermite configuration



weight (x) (-1)

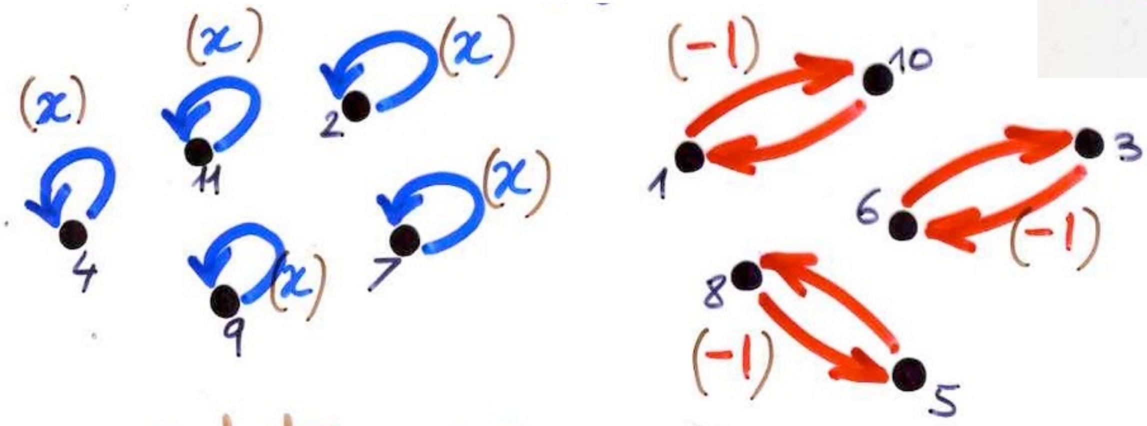
$$\sigma^2 = Id$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 2 & 6 & 4 & 8 & 3 & 7 & 5 & 9 & 1 & 11 \end{pmatrix}$$



$\mathfrak{S}_n$ symmetric group

Hermite configuration



weight (x)

(-1)

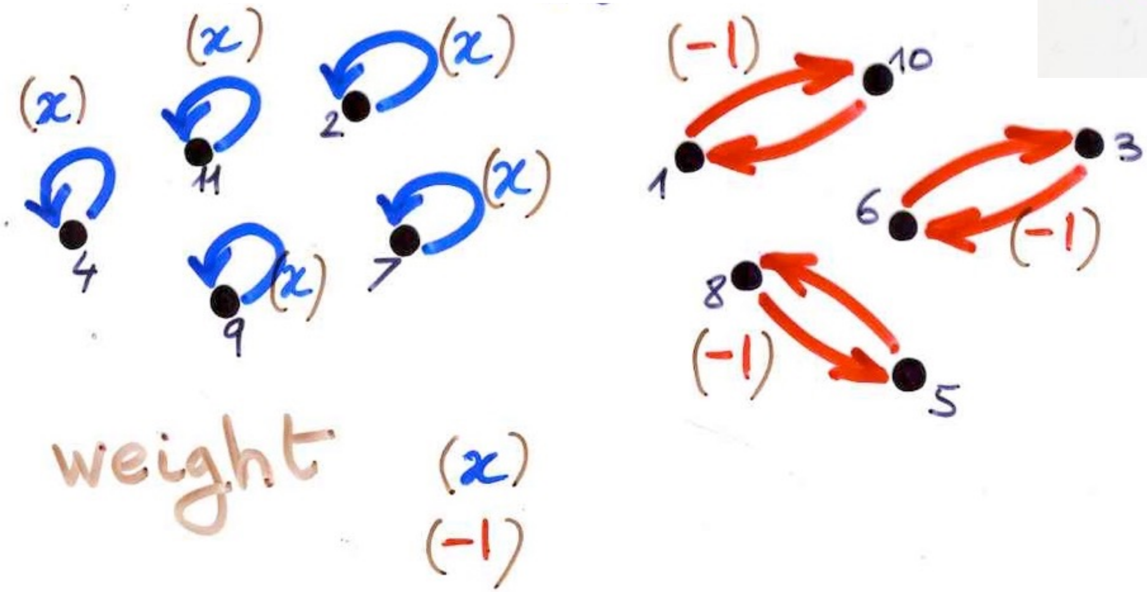
$d(\sigma)$ = number of cycles of σ of length 2

$\mathrm{fix}(\sigma)$ = number of fixed points of σ

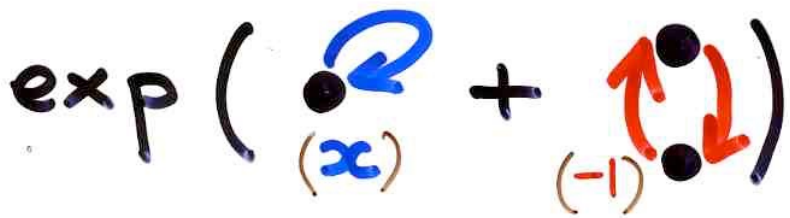
$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$

(combinatorial) Hermite polynomials

Hermite configuration



(combinatorial) Hermite polynomials



$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

Chapter 5 Orthogonality and exponential structures

exercise

Hermite polynomials

$$H_n(x) = \sum_{0 \leq 2k < n} (-1)^k \frac{n!}{2^k\, k!\,(n-2k)!}\, x^{n-2k}$$

exercise

3-terms linear recurrence relation

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

Laguerre polynomials

Laguerre polynomial

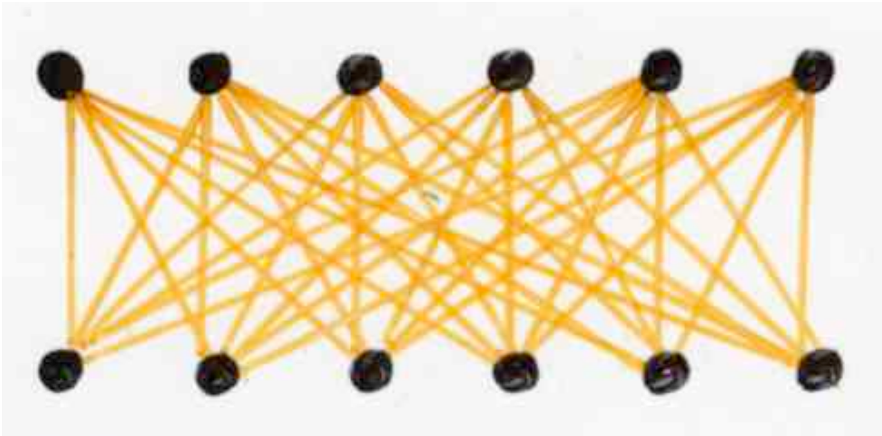
$L_n(x)$

$L_n^{(\alpha)}(x)$ $\alpha = 0$

Laguerre polynomial $L_n(x)$ = matching polynomial of $K_{n,n}$

complete bipartite graph

$K_{6,6}$

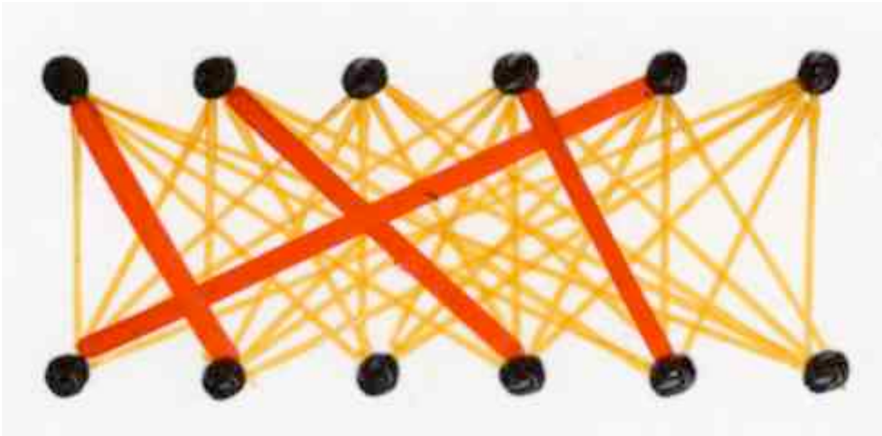


Laguerre polynomial
$L_n(x)$

= matching polynomial of $K_{n,n}$

complete
bipartite
graph

$K_{6,6}$



exercise

$$L_n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \binom{n}{k} x^k$$

usual Laguerre polynomials

$$\frac{(-1)^n}{n!} L_n(x)$$

exercise

Laguerre polynomials

$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$

3-terms linear recurrence relation

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

(formal) orthogonal polynomials

$$f\big(P(x)\,Q(x)\big) = \int_{\mathbb{R}} P(x)\,Q(x)\,d\mu(x)$$

measure μ on $\mathbb{R}$

$$f(x^n) = \int_{\mathbb{R}} x^n\,d\mu(x)$$

moments problem

$$f(x^n) = \mu_n$$

moments

$\mathbb{K}$ ring

field $\mathbb{R}$, $\mathbb{C}$
or $\mathbb{Q}[\alpha, \beta, \ldots]$

$\mathbb{K}[x]$
polynomials in x

$\{P_n(x)\}_{n\geq 0}$
sequence of polynomials

$P_n(x) \in \mathbb{K}[x]$.

Definition $\{P_n(x)\}_{n\geq 0}$ sequence of polynomials orthogonal iff $\exists$

$\mathfrak{f} : \mathbb{K}[x] \longrightarrow \mathbb{K}$ linear functional

(i) $\deg(P_n) = n$, for $n \geq 0$

degree

(ii) $\mathfrak{f}(P_k P_\ell) = 0$, for $k \neq \ell \geq 0$

(iii) $\mathfrak{f}(P_k^2) \neq 0$, for $k \geq 0$

$$\mathfrak{f}(x^n) = \mu_n$$

moments

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

moments of (Tchebychev) 1 st kind
2 nd kind

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Catalan number

$$\frac{2}{\pi} \int_{-1}^{+1} x^{2n} (1-x^2)^{1/2} \, dx = \frac{1}{4^n} C_n$$

Catalan

moments of Hermite polynomial

$$\mu_{2n+1} = 0$$

$$\mu_{2n} = 1 \times 3 \times \dots \times (2n-1)$$

number of involutions on $\{1, \dots, 2n\}$ with no fixed points

moments Laguerre polynomials

$$\mu_n = n!$$

First steps with sign-reversing involutions

Orthogonality of Hermite
Laguerre

Combinatorial interpretation of Linearization coefficients

linearization coefficients
and orthogonality

exemple: Hermite polynomials

linearization coefficients

Lemma

$$P_k(x)\,P_\ell(x) = \sum_n a_{k\ell}^n\, P_n(x)$$

$$a_{k\ell}^n = \frac{f(P_k P_n P_\ell)}{f(P_n^2)}$$

$$f\big(H_{n_1}(x)\, H_{n_2}(x) \cdots H_{n_k}(x)\big)$$

$$\mathbf{f}(x^n) = \mu_n$$

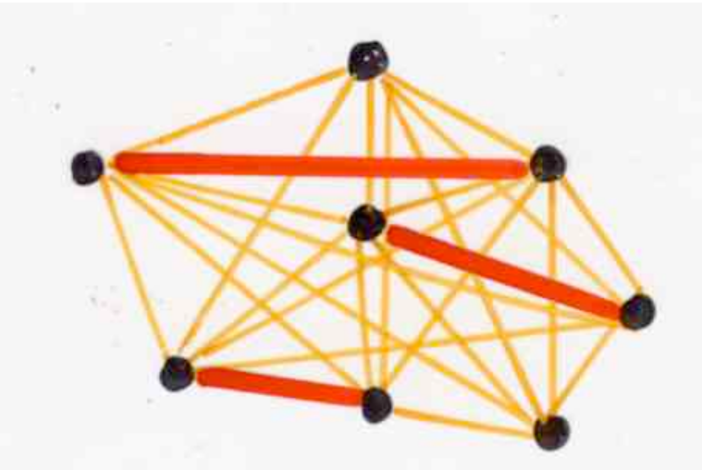
moments

moments of Hermite polynomial

$$\mu_{2n+1} = 0$$
$$\mu_{2n} = 1 \times 3 \times \dots \times (2n-1)$$

exercise

number of involutions on $\{1, \dots, 2n\}$ with no fixed points $= 1 \times 3 \times \dots \times (2n-1)$



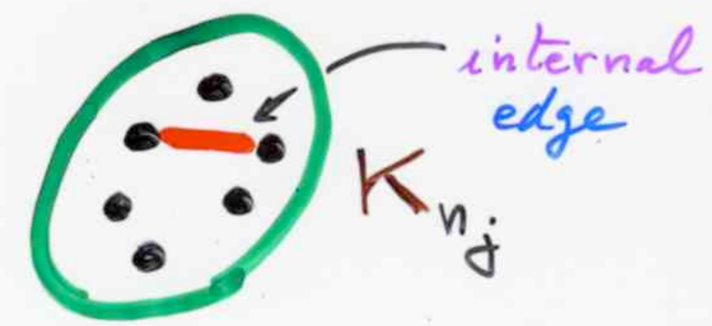
$H_n(x)$

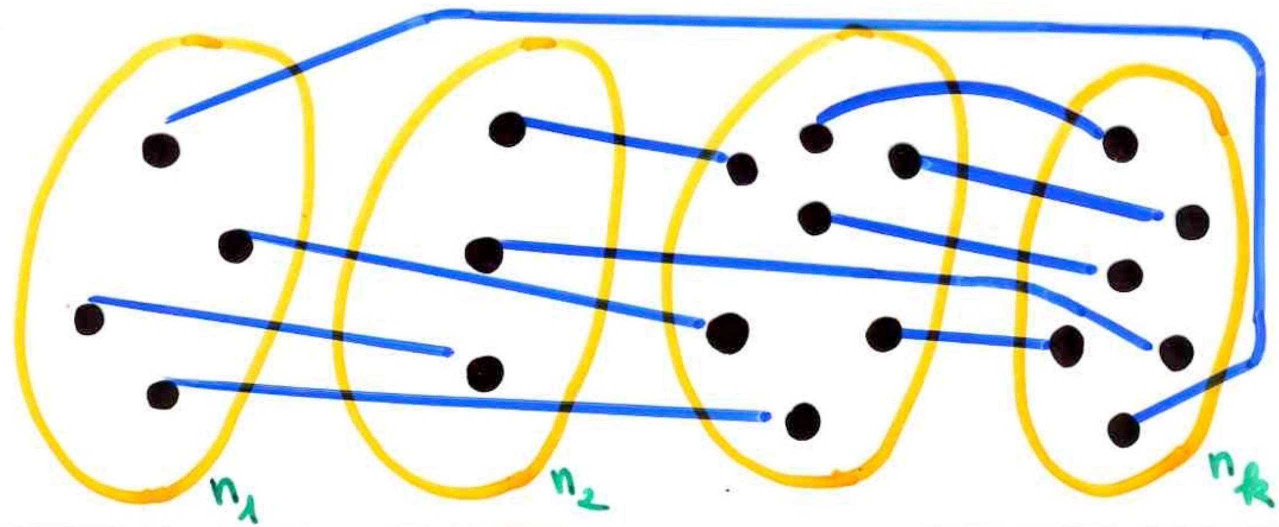
matching polynomial of the complete graph K_n

Proposition

$$\mathfrak{f}\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right) =$$

number of perfect matchings of the graph $K_{n_1} \oplus K_{n_2} \oplus \cdots \oplus K_{n_k}$ with no "internal" edges



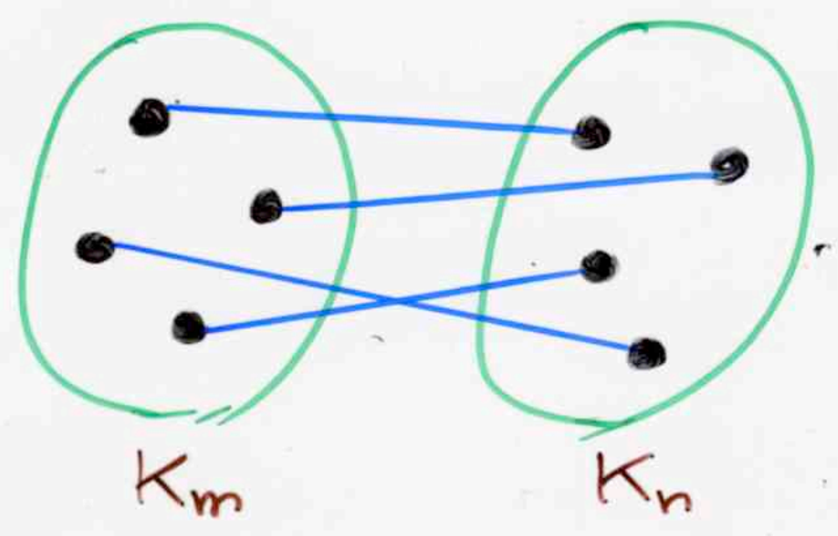


in particular:

Corollary

orthogonality!

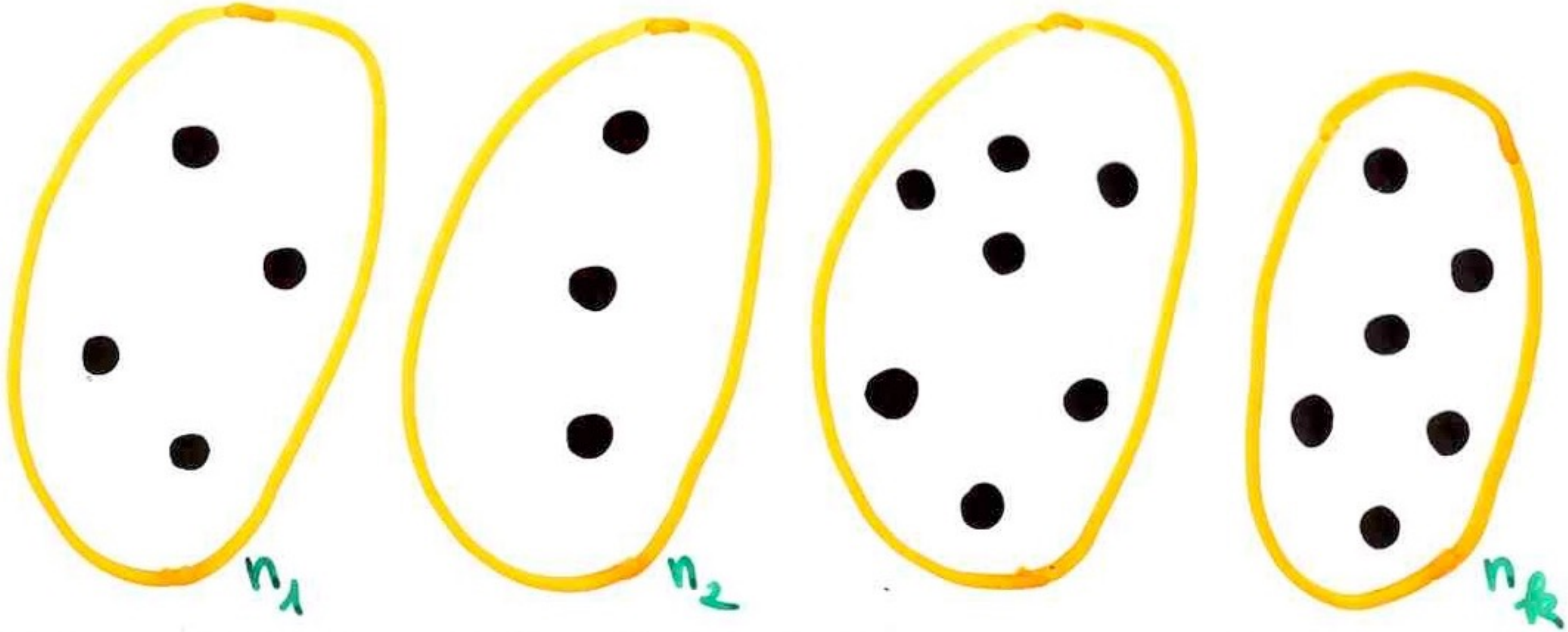
$$\mathcal{L}\big(H_m(x)\,H_n(x)\big) = n!\,\delta_{m,n}$$



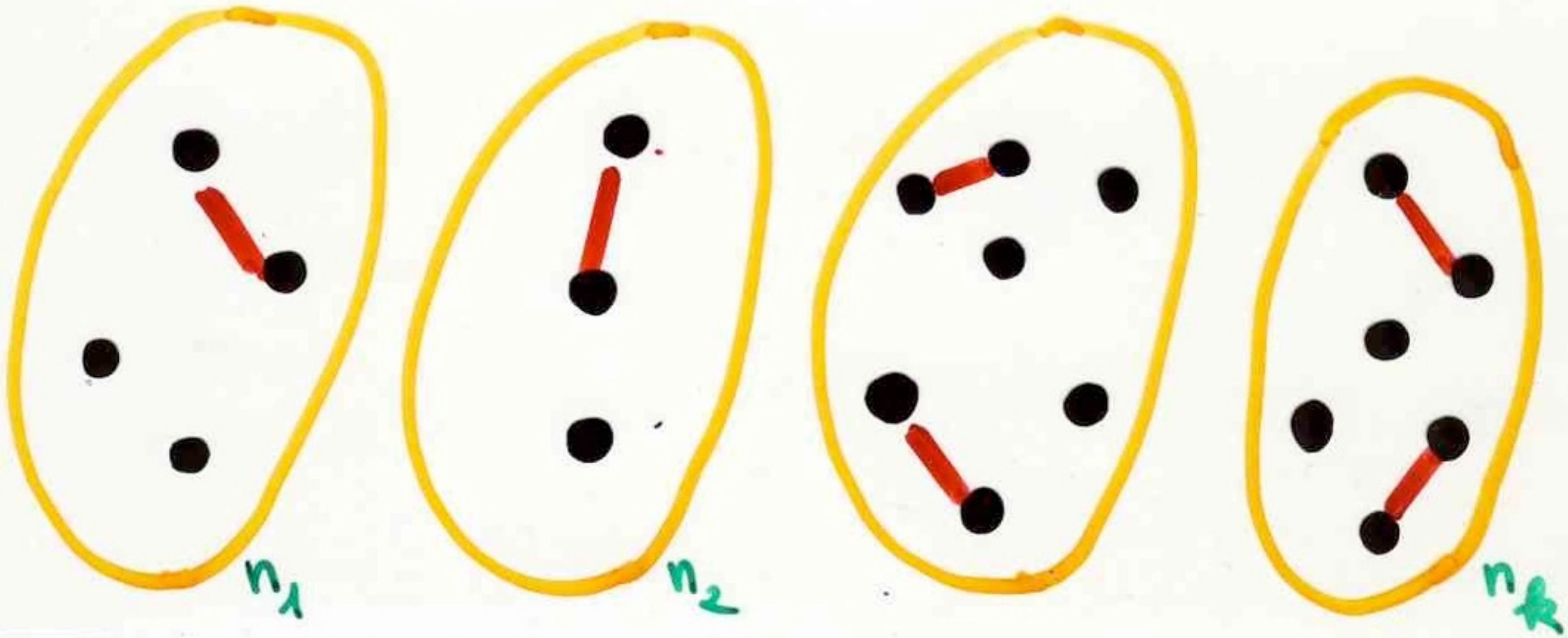
$$H_{n_1}(x)\, H_{n_2}(x) \cdots H_{n_k}(x)$$

$$= \sum_{\alpha_1, \dots, \alpha_k} (-1)^{|\alpha_1| + \cdots + |\alpha_k|} \, x^{ip(\alpha_1) + \cdots + ip(\alpha_k)}$$

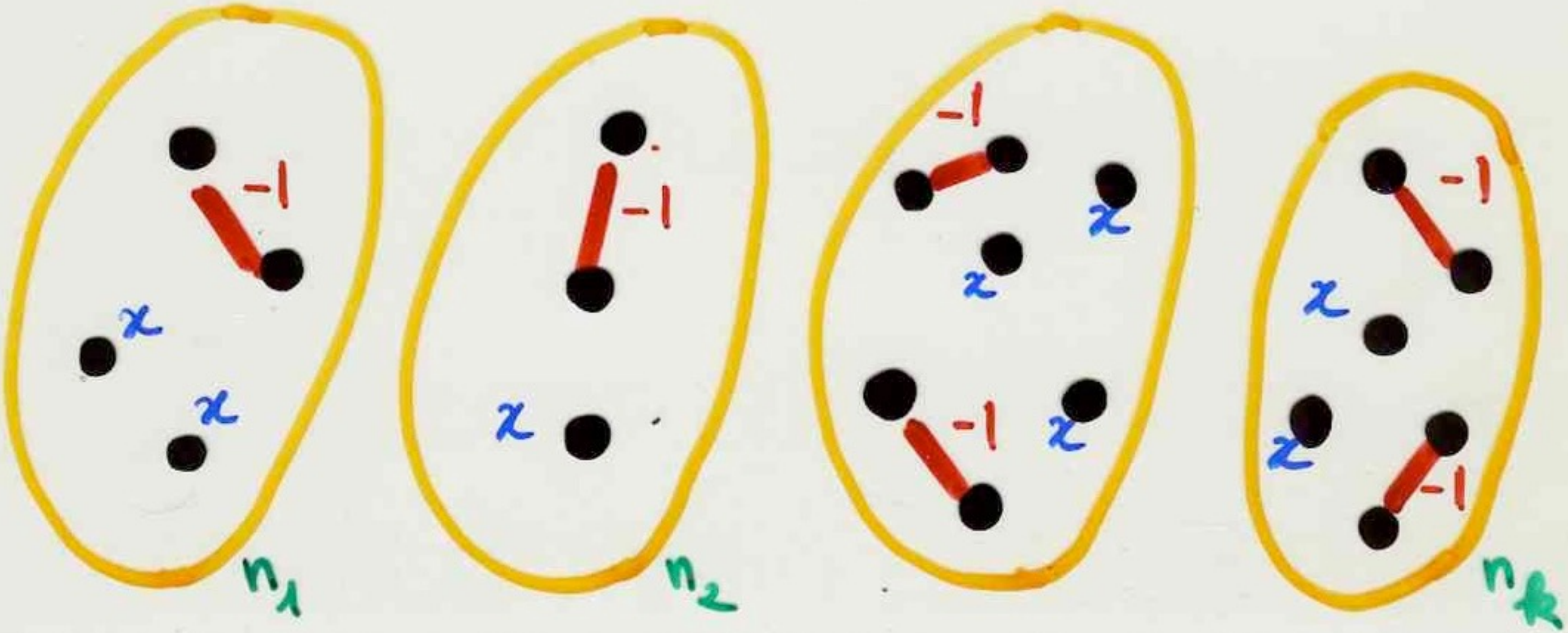
α_i matching of K_i


n_1
n_2
n_k

$$H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)$$



$$H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)$$



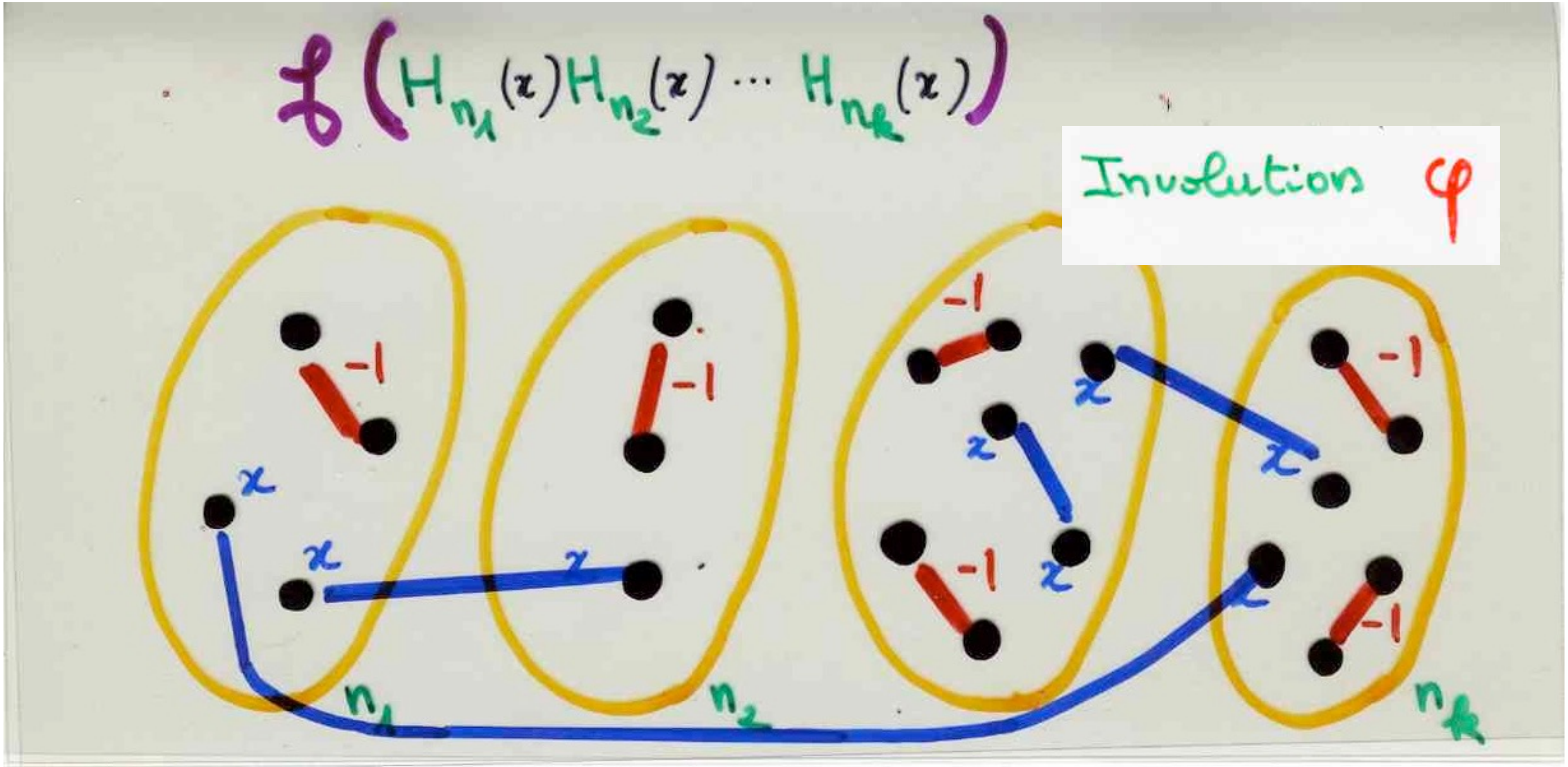
$$H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)$$

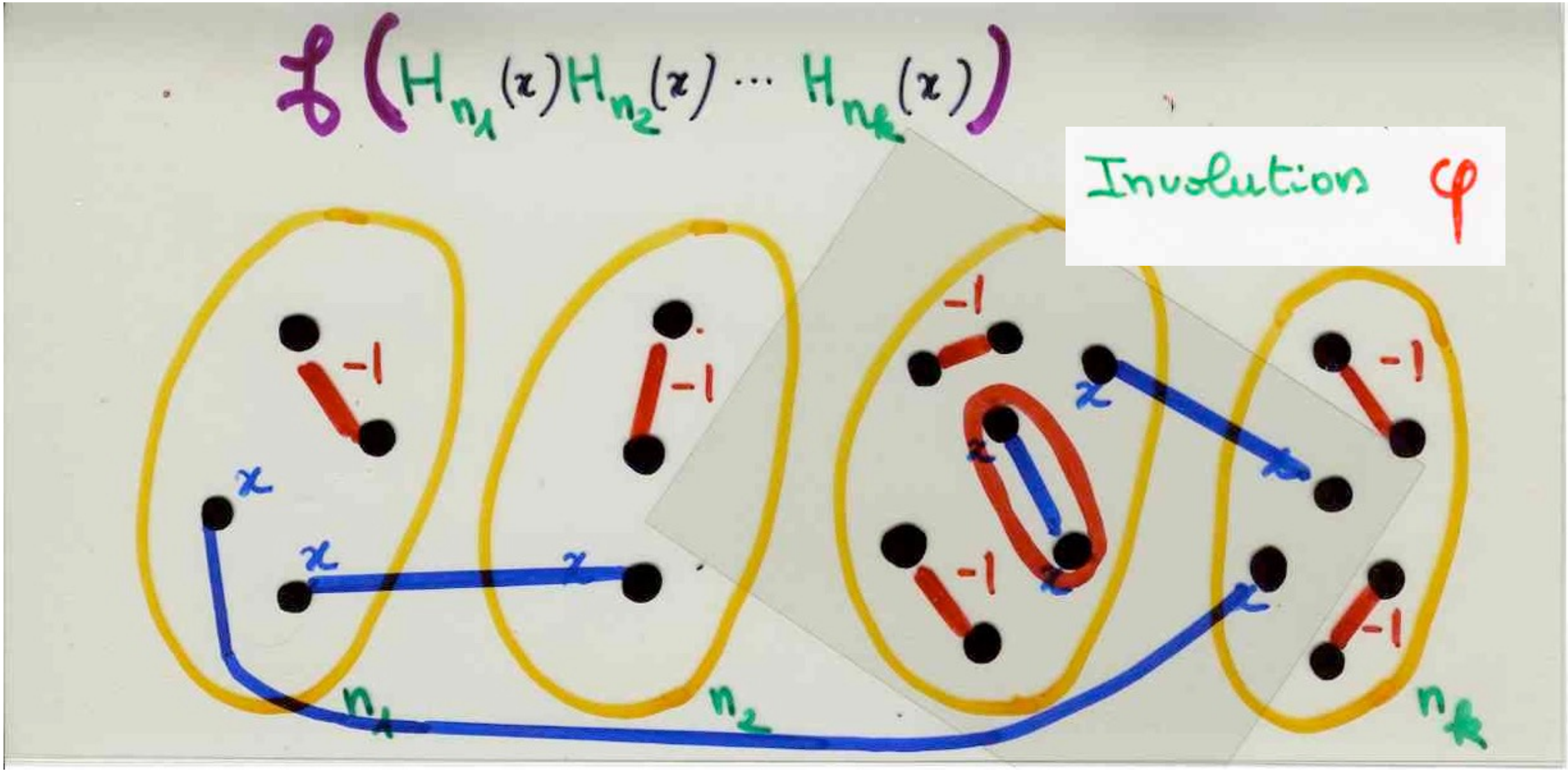
$$= \sum_{\substack{\alpha_1,\dots,\alpha_k \\ \alpha_i \text{ matching of } K_i}} (-1)^{|\alpha_1|+\cdots+|\alpha_k|}\, x^{ip(\alpha_1)+\cdots+ip(\alpha_k)}$$

$$f\big(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\big)$$

$$= \sum_{\substack{(\alpha_1,\dots,\alpha_k\,;\,\alpha) \\ \alpha_i \text{ matching of } K_i \\ \alpha \text{ perfect matching on } K(\alpha_1,\dots,\alpha_k)}} (-1)^{|\alpha_1|+\cdots+|\alpha_k|}$$

the complete graph with vertices
= the set of isolated points of $\alpha_1,\dots,\alpha_k$


$\mathcal{L}\left(H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)\right)$
Involution φ
-1
x
n_1
n_2
n_k

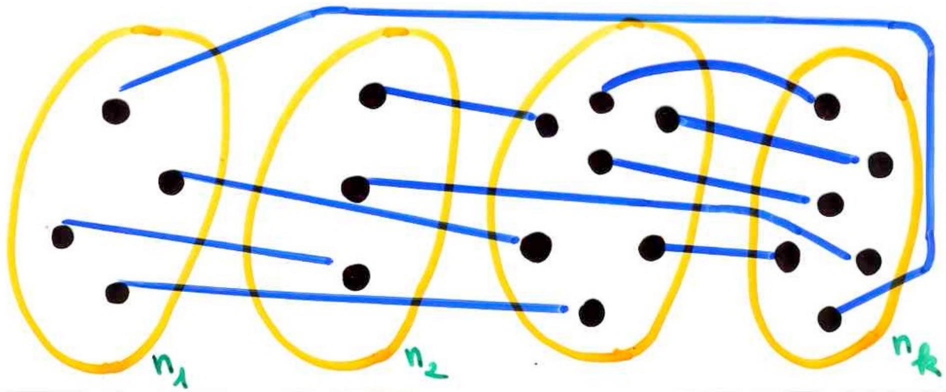

$\mathcal{L}\left(H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)\right)$
Involution φ
-1
x
n_1
n_2
n_k

Involution φ

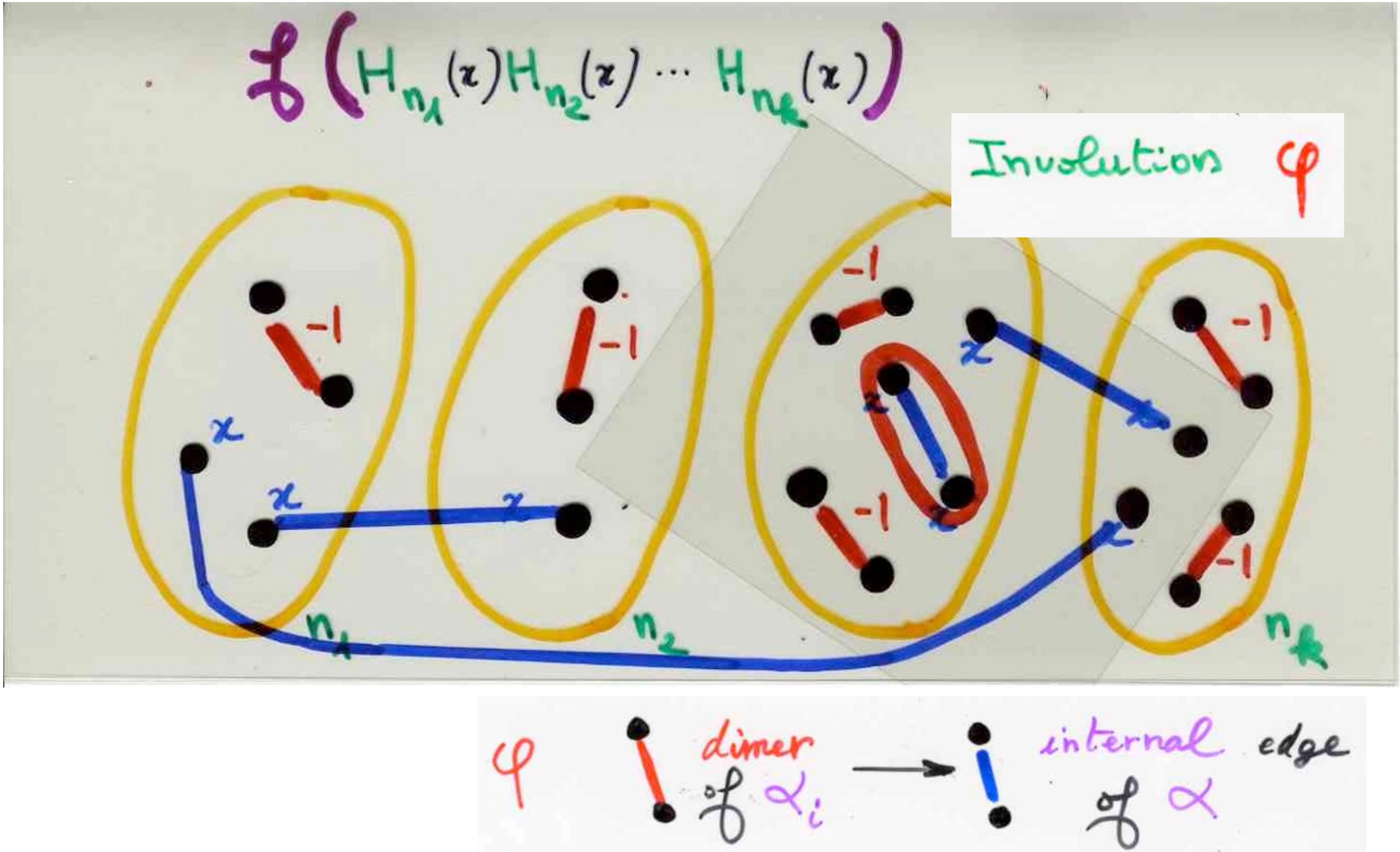
not defined on the set

$(\alpha_1 = \emptyset, \dots, \alpha_k = \emptyset \,;\, \alpha)$

α with no "internal" edge



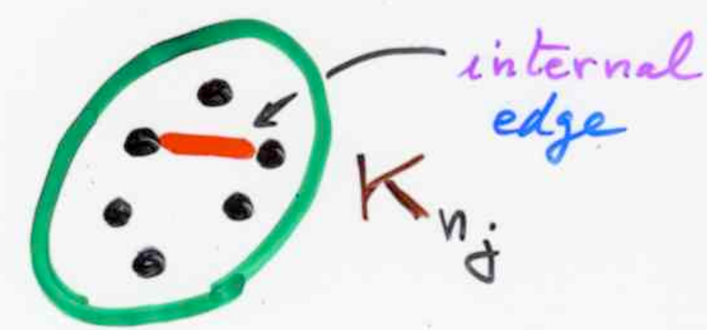
φ dimer of α_i $\longrightarrow$ internal edge of α

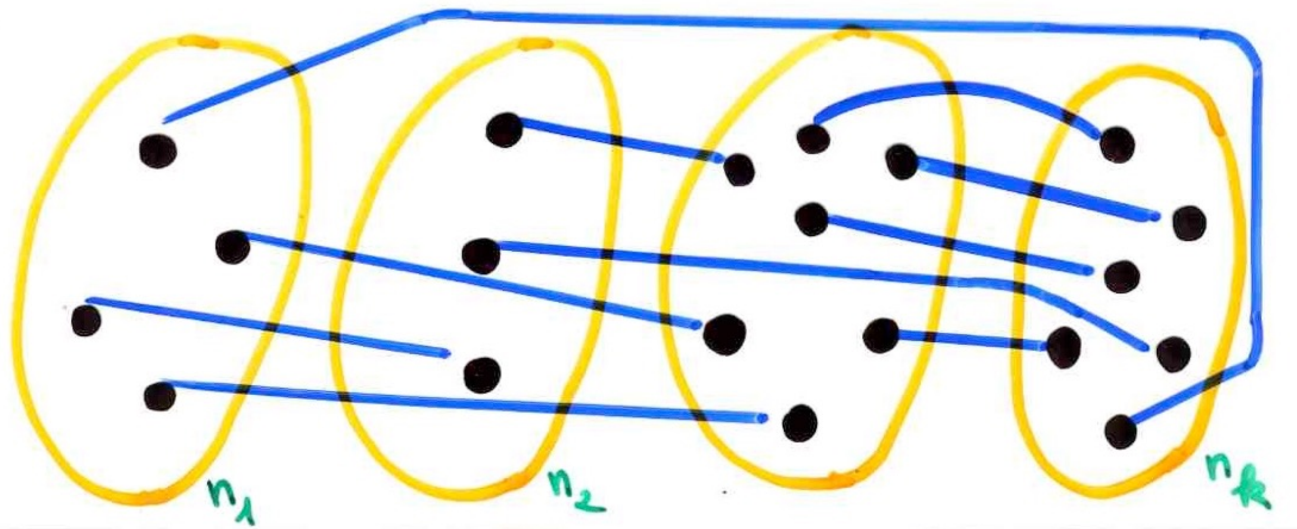


f (Hn1(x)Hn2(x) ··· Hnk(x))
Involution φ
-1
x
n1
n2
nk
φ dimer of αi → internal edge of α

$$\mathcal{f}\left(H_{n_1}(x)\, H_{n_2}(x) \cdots H_{n_k}(x)\right) =$$

number of perfect matchings of the graph $K_{n_1} \oplus K_{n_2} \oplus \cdots \oplus K_{n_k}$ with no "internal" edges



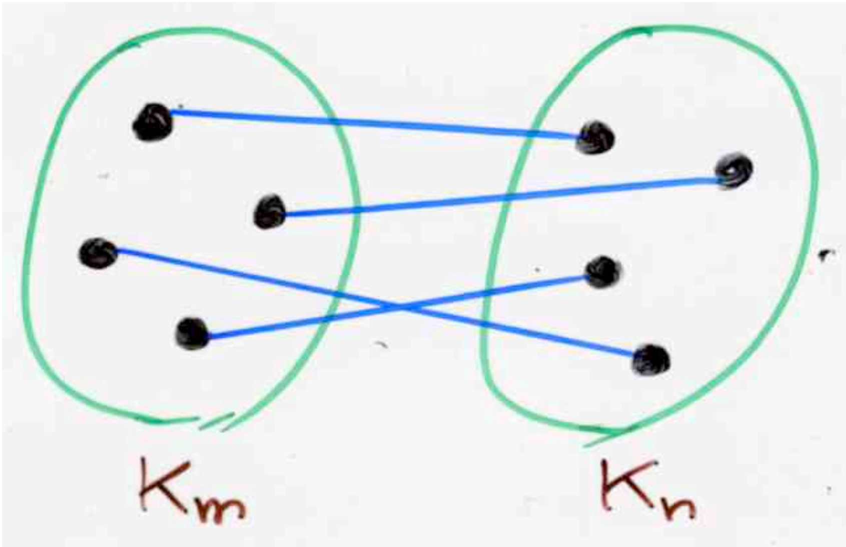


$\square$

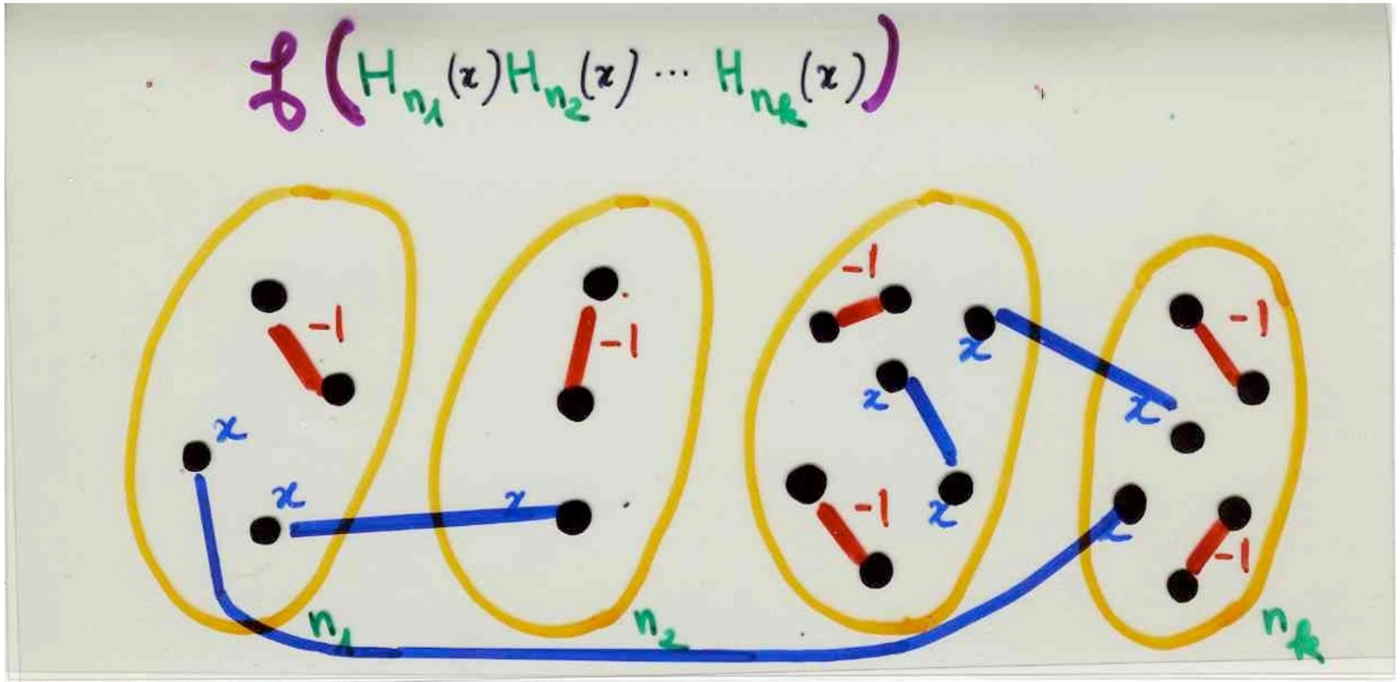
in particular:

orthogonality!

$$\mathcal{f}\left(H_m(x)\, H_n(x) \right) = n!\, \delta_{m,n}$$



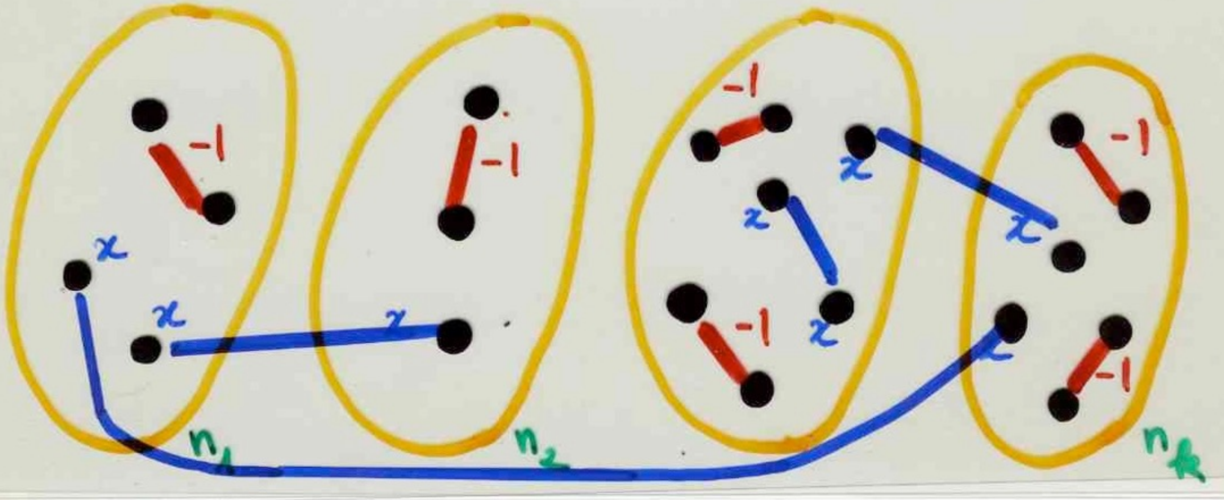
another "sign-reversing" proof
without explicit construction
of an involution



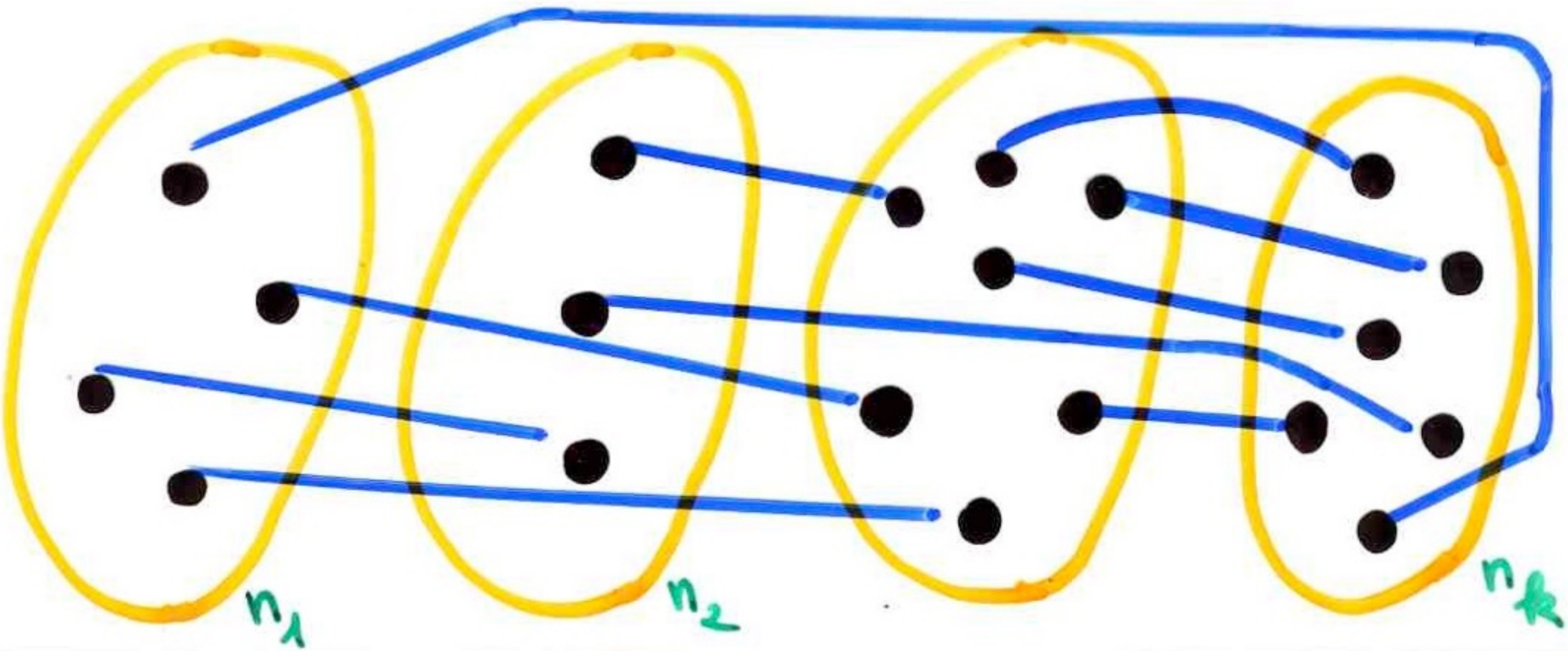
$$\sum_{\substack{(\alpha_1,\dots,\alpha_k;\,\alpha) \\ [\text{edges of } \alpha_1,\dots,\alpha_k] \,\cup\, [\text{internal edges of } \alpha] \,=\, F}} (-1)^{|\alpha_1|+\dots+|\alpha_k|} = \sum_{e\in F} \left[1+(-1)\right] = 0$$

$2^{|F|}$ terms

$$f\left(H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)\right)$$



$$\mathcal{L}\left(\mathbf{H}_{n_1}(x)\,\mathbf{H}_{n_2}(x)\cdots\mathbf{H}_{n_k}(x)\right) =$$

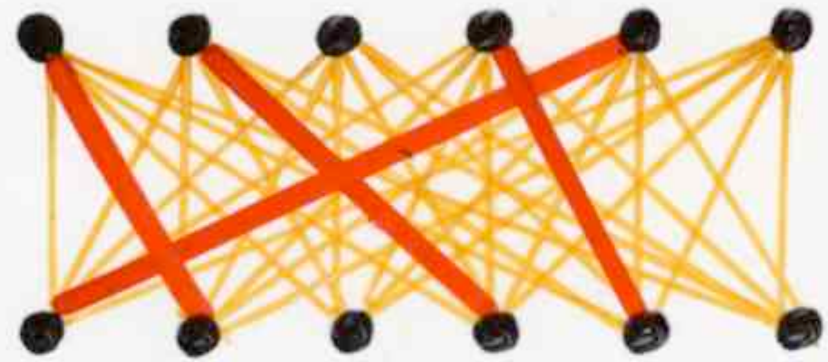


linearization coefficients
and orthogonality

exemple: Laguerre polynomials

Laguerre polynomial $L_n(x)$ = matching polynomial of $K_{n,n}$

complete bipartite graph



$f(x^n) = \mu_n$ moments

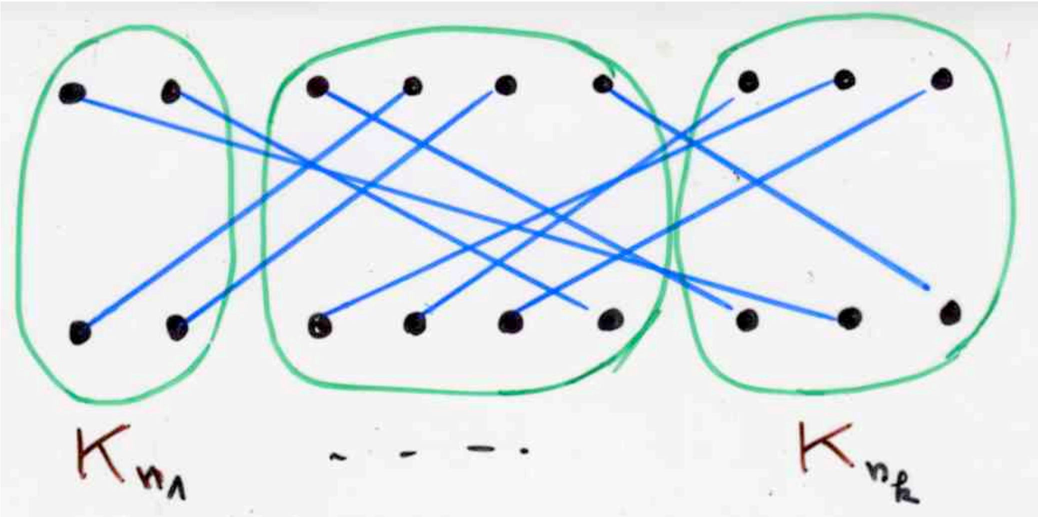
$\mu_n = n!$

moments Laguerre polynomials

Proposition exercise

$$\mathcal{f}\left(L_{n_1}(x)\, L_{n_2}(x) \cdots L_{n_k}(x)\right) =$$

number of perfect matchings of the graph L $K_{n_1,n_1} \oplus \cdots \oplus K_{n_k,n_k}$ with no "internal" edges



in particular:

Corollary orthogonality !

$$f\left(L_m(x) L_n(x) \right) = (n!)^2 \, \delta_{m,n}$$