Colloque PFAC

Philippe Flajolet and analytic combinatorics

Combinatorial aspects of continued fractions and applications

Paris,

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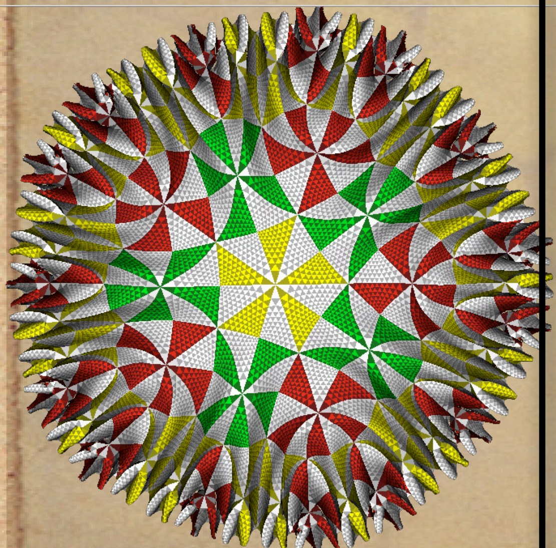
Happy New Year 2009



GIFT. *Define the "equiharmonic numbers" by*

$$K_\nu := \frac{(6\nu)!}{\Omega^{6\nu}} \sum_{(n_1,n_2)\in(\mathbb{Z}\times\mathbb{Z})\setminus\{(0,0)\}} \frac{1}{(n_1 e^{-2i\pi/3} + n_2 e^{2i\pi/3})^{6\nu}}, \qquad \Omega := \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)^3.$$

The generating function of (K_ν) *admits the continued fraction representation*

$$\frac{7}{36}\sum_{\nu\geq 1} K_\nu z^{\nu-1} = \cfrac{1}{1-\cfrac{d_1\cdot z}{1-\cfrac{d_2\cdot z}{\ddots}}},$$

where $d_1 = \frac{10880}{13}$, $d_2 = \frac{13810240}{247}$, $d_n = \frac{1}{4}\frac{(3n)(3n+1)^2(3n+2)^2(3n+3)^2(3n+4)}{(6n+1)(6n+7)}$.

27 Dec 2008

ϕ nombre d'or $\frac{1+\sqrt{5}}{2}$

$$t^2 - t - 1 = 0$$

$$\phi - 1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

$$\begin{array}{cccccc} F_0 & F_1 & F_2 & F_3 & F_4 & F_5 \,,\ \ldots \\ 1 & 1 & 2 & 3 & 5 & 8 \,,\ \ldots \end{array}$$

Fibonacci

$k^{\text{ième}}$ convergent

$$\left.\cfrac{1}{1+\cfrac{1}{1+\cdots\cfrac{1}{1+1}}}\right\}k = \frac{F_k}{F_{k+1}}$$

arithmetical continued fractions

Apéry

$$\zeta(3) = \sum 1/n^3$$

irrational

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\cdots}}}}$$

$$\varpi(n) = (2n+1)(17n(n+1)+5)$$

analytic continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{1-b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$$J(t; b, \lambda)$$

Jacobi continued fraction

$$b = \{b_k\}_{k\geq 0} \qquad \lambda = \{\lambda_k\}_{k\geq 1}$$

$$\sum_{n\geq 0} n!\, t^n =$$

$$\cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots\cdots}}}}$$

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cfrac{\cdots\cdots}{1-\cfrac{\lambda_k t}{\cdots}}}}}}_{S(t;\lambda)}$$

$\mu_0 = 1$

Stieltjès continued fraction

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}}}$$

§. 21. Datur vero alius modus in summam huius seriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: sit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

AVCTORE

Leonh. Euler.

§. 1.

Varii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4\ 4.6.\ 6.8.\ 8.10.\ 10}{3.3.\ 5.5.\ 7.7\ \ 9.9.\ \ 11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

The fundamental Flajolet Lemma

Theorem and Lemma
«Proof from the book»
Aigner, Ziegler

The fundamental Flajolet Lemma

combinatorial interpretation of a continued fraction with weighted paths

Discrete Mathematics 32 (1980) 125–161

Discrete Maths (1980)

COMBINATORIAL ASPECTS OF CONTINUED FRACTIONS

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Received 23 March 1979
Revised 11 February 1980

We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers.... We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

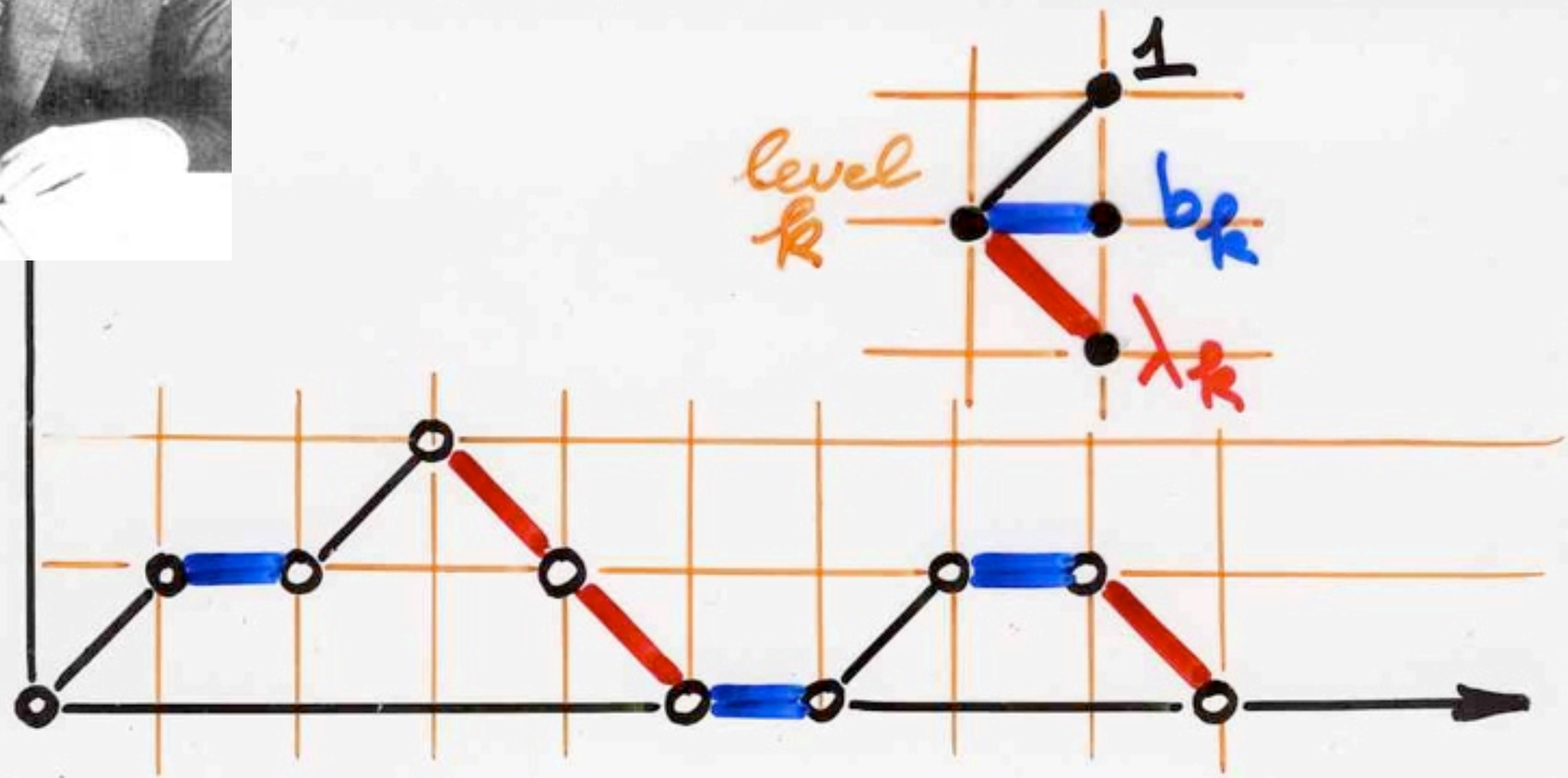
Introduction

In this paper we present a geometrical interpretation of continued fractions together with some of its enumerative consequences. The basis is the equivalence

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

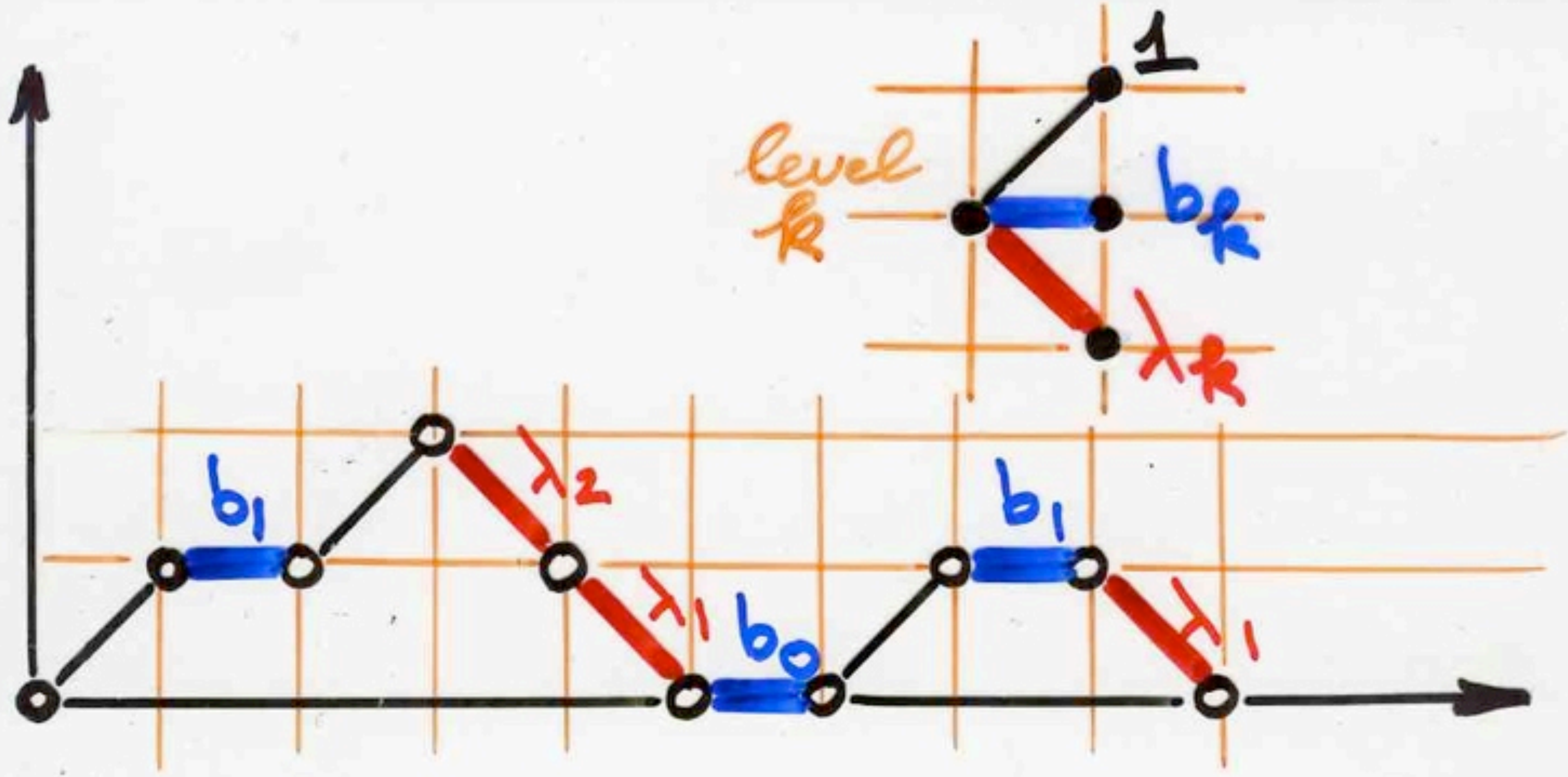
$b_k, \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cfrac{\cdots}{\cdots}}}}}}$$

Philippe Flajolet
fundamental
Lemma

Jacobi continued fraction

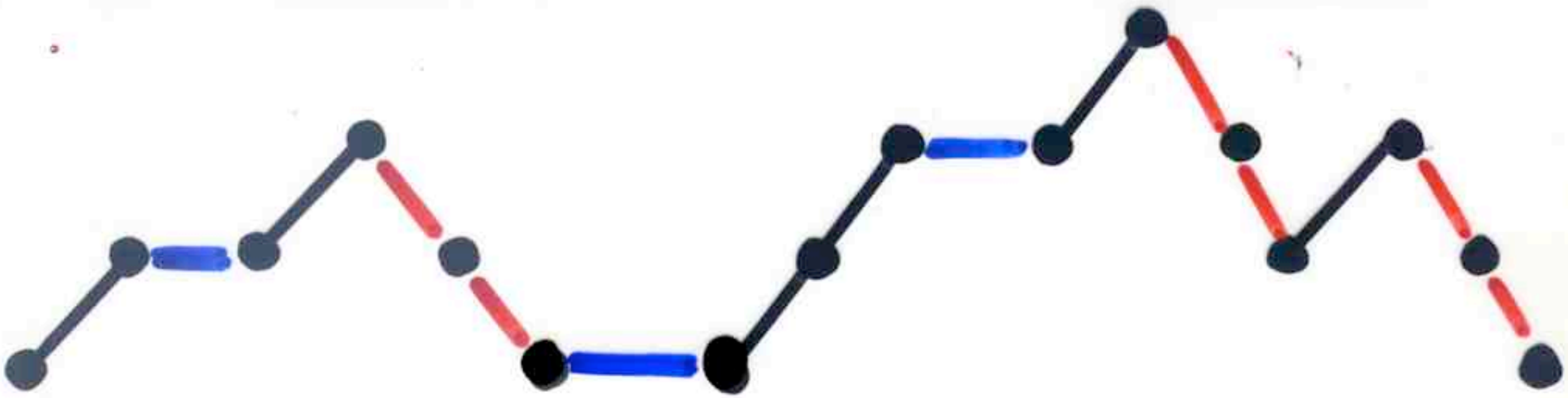
$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}}$$

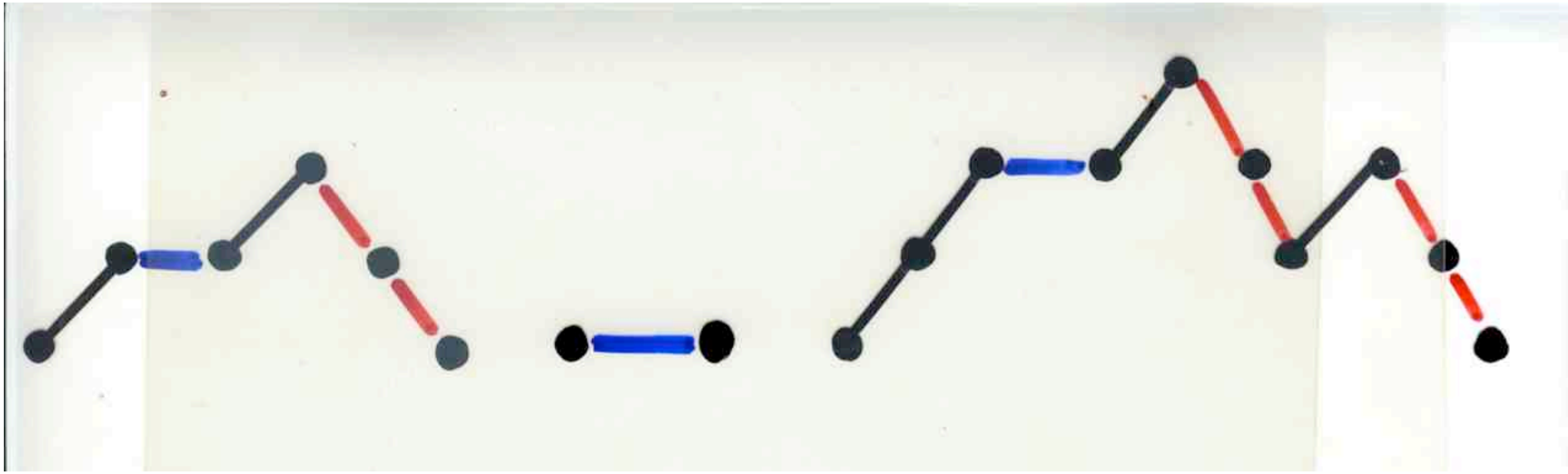
$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega|=n}} v(\omega)$$

Philippe Flajolet fundamental

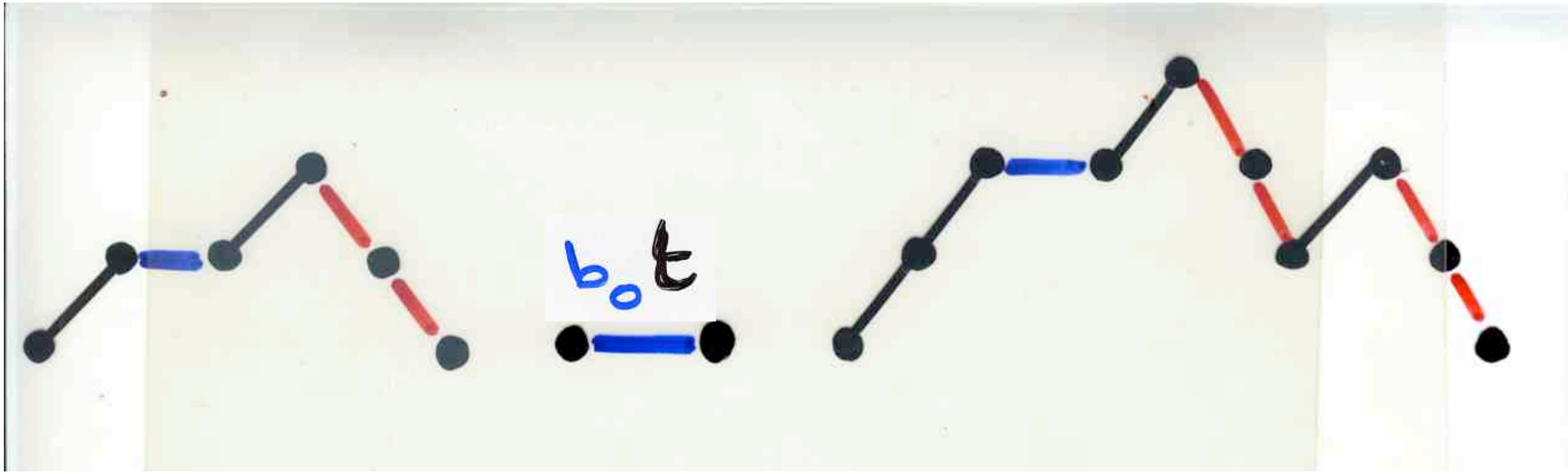
Lemma

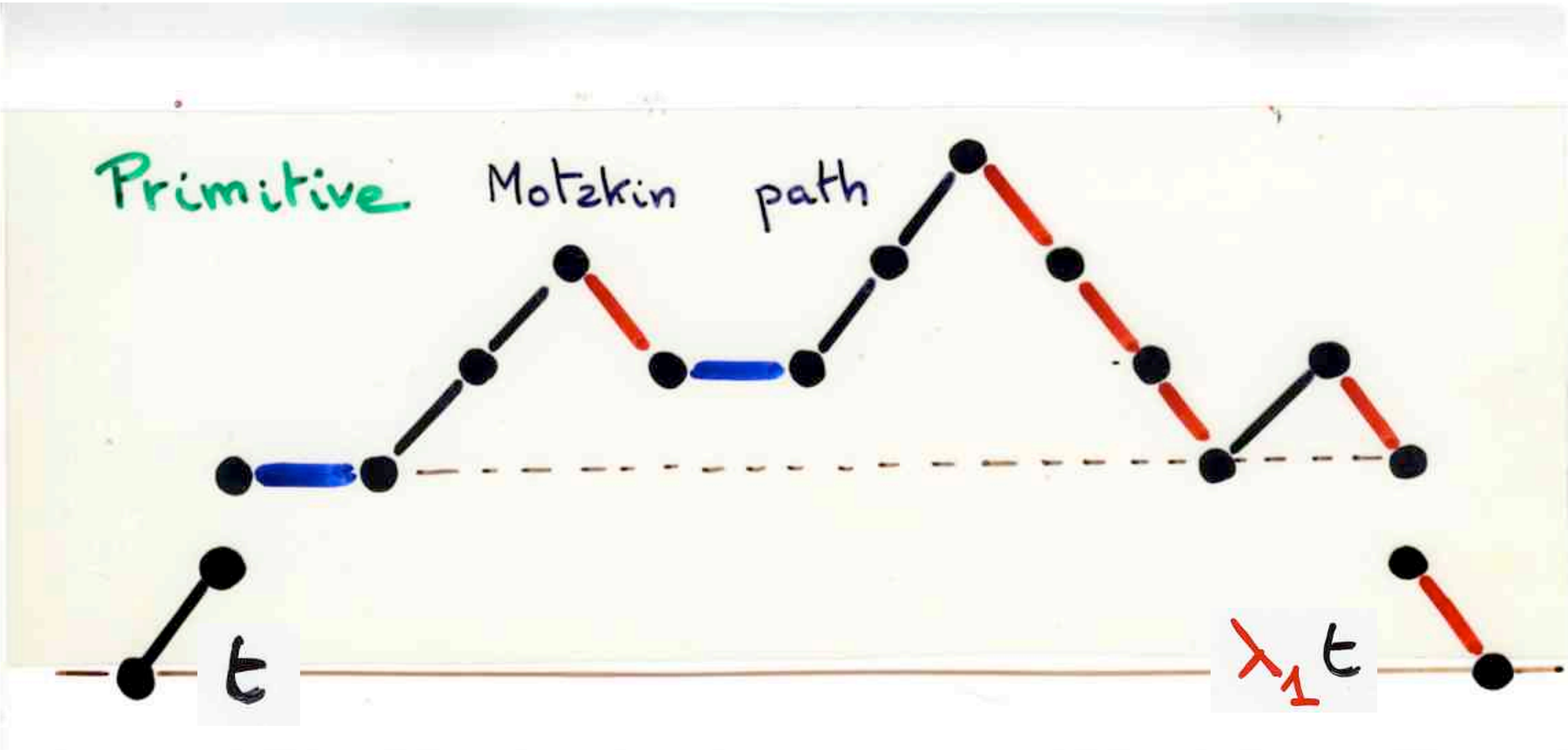
proof:





$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \frac{1}{1 - \displaystyle\sum_{\substack{\omega \\ \text{primitive} \\ \text{Motzkin} \\ \text{path}}} v(\omega)}$$


$b_0 t$

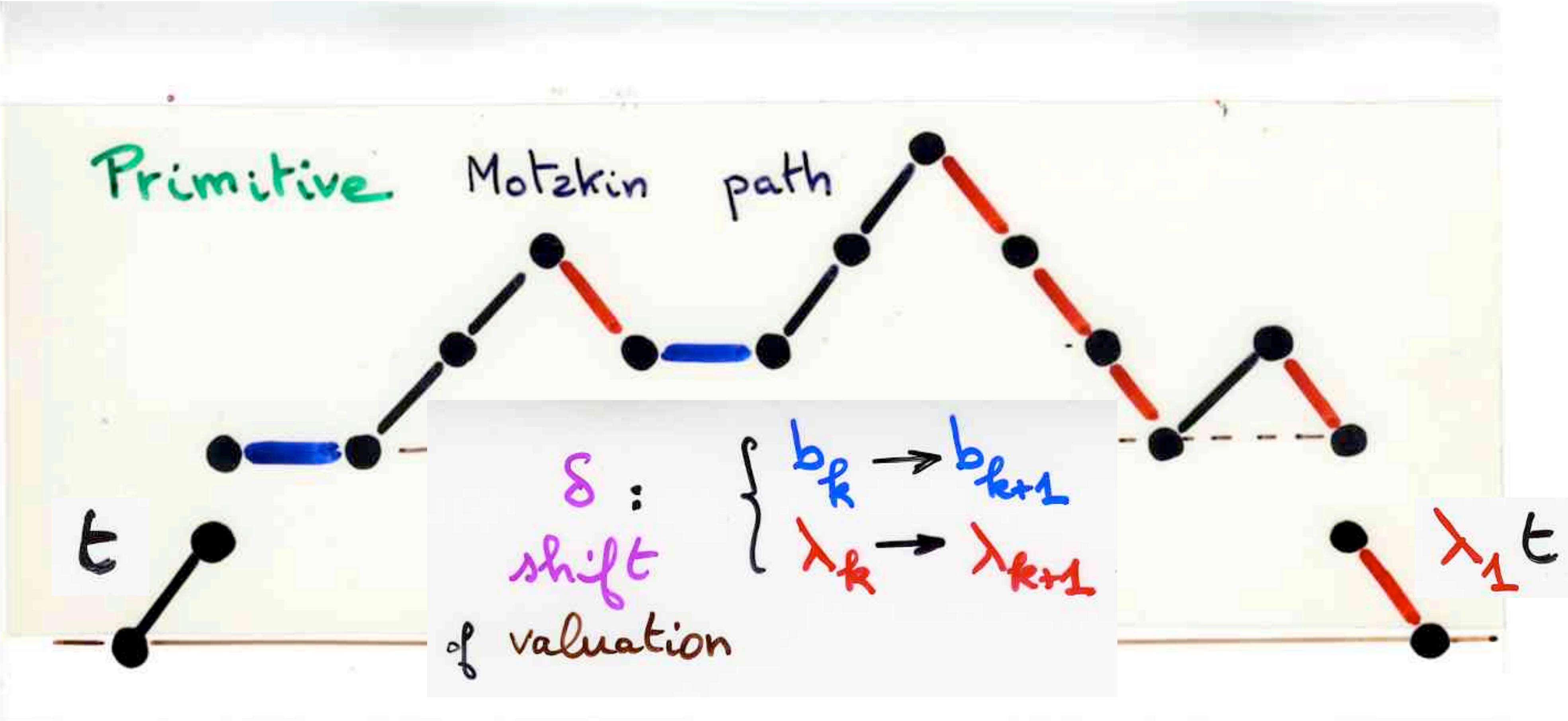

Primitive Motzkin path
t
$\lambda_1 t$

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \frac{1}{1 - b_0 t - \lambda_1 t^2} \quad (\text{same})$$

δ : $\left\{ \begin{array}{l} b_k \to b_{k+1} \\ \lambda_k \to \lambda_{k+1} \end{array} \right.$

shift

of valuation



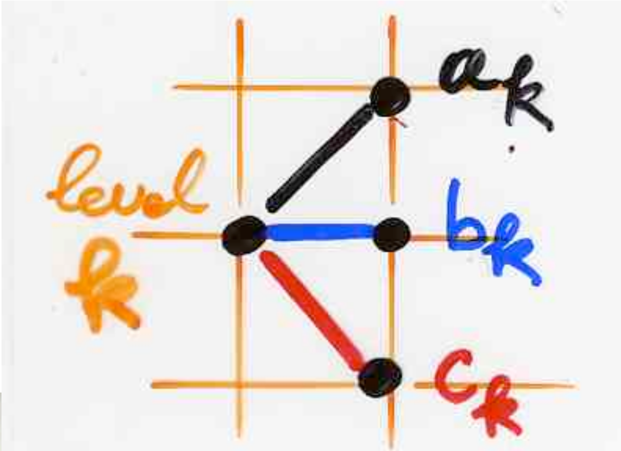
Primitive Motzkin path
t
δ : shift of valuation
$b_k \to b_{k+1}$
$\lambda_k \to \lambda_{k+1}$
$\lambda_1 t$

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \lambda_2 t^2 \underbrace{(\,''\,)}_{\delta^2}}}$$

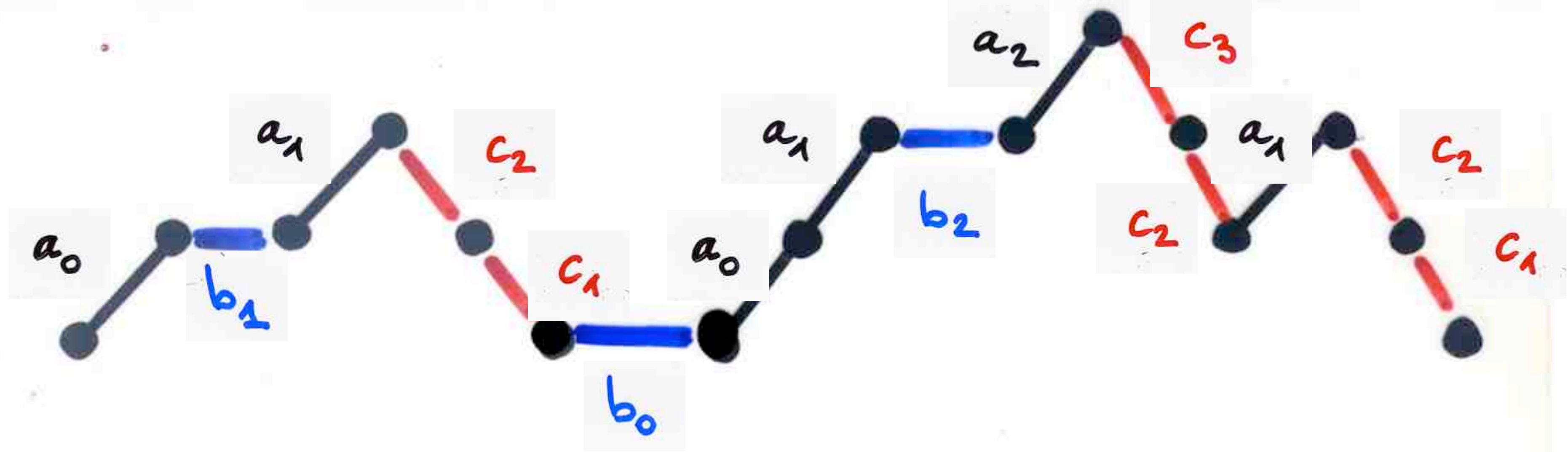
Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cfrac{\cdots}{\cdots}}}}}}$$

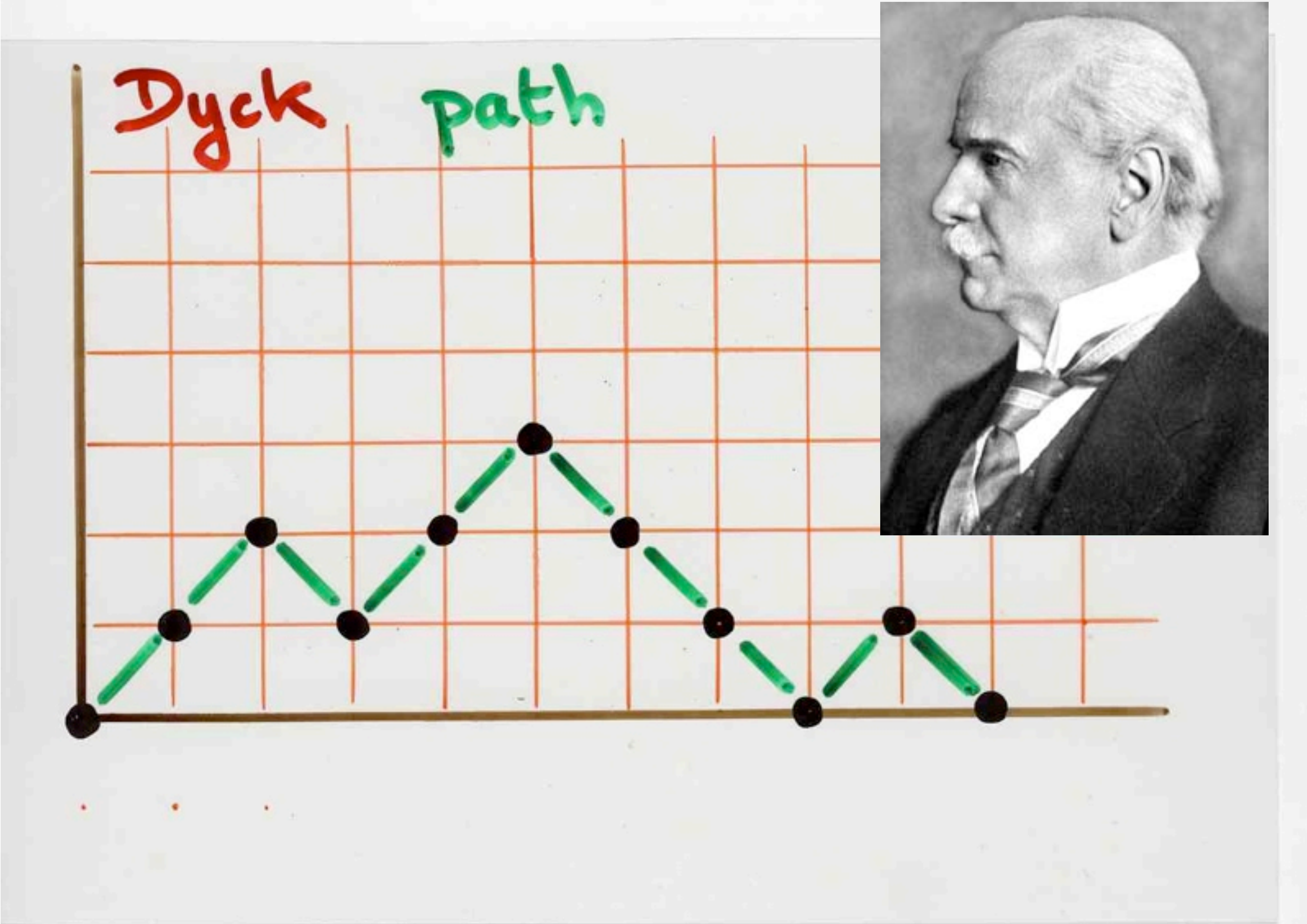
Philippe Flajolet
fundamental
Lemma

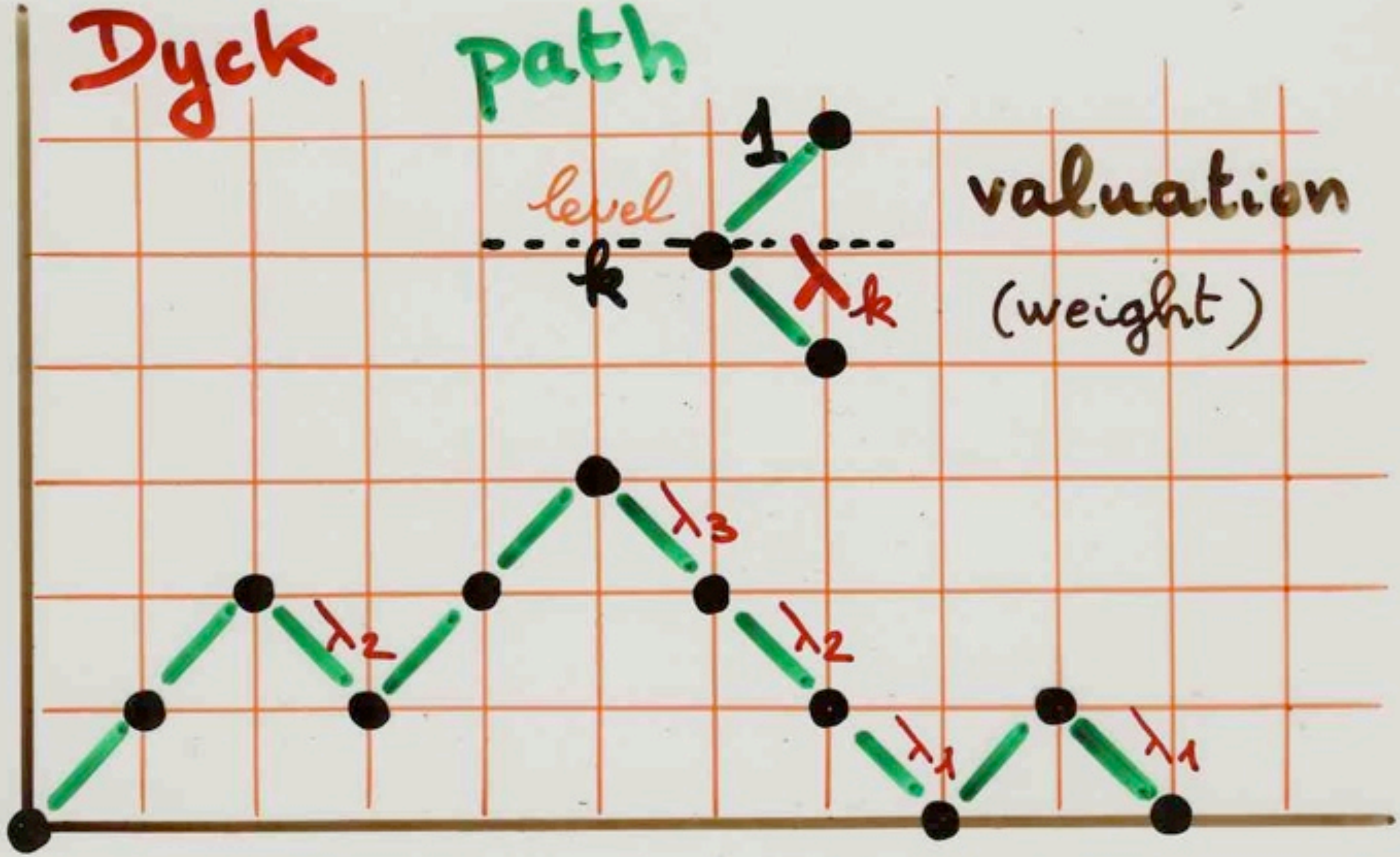


non-commutative
power series



$$\left[b_0 + a_0 \left(b_1 + a_1 (\cdots\cdots)^{*} c_2 \right)^{*} c_1 \right]^{*}$$


Dyck path



weight $v(\omega) = \lambda_1^2 \, \lambda_2^2 \, \lambda_3$

continued fractions

$$\sum_{\substack{\omega \\ \text{Dyck} \\ \text{path}}} v(\omega)\, t^{|\omega|/2} = \underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots \cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}}_{S(t;\lambda)}$$

Stieltjes continued fraction

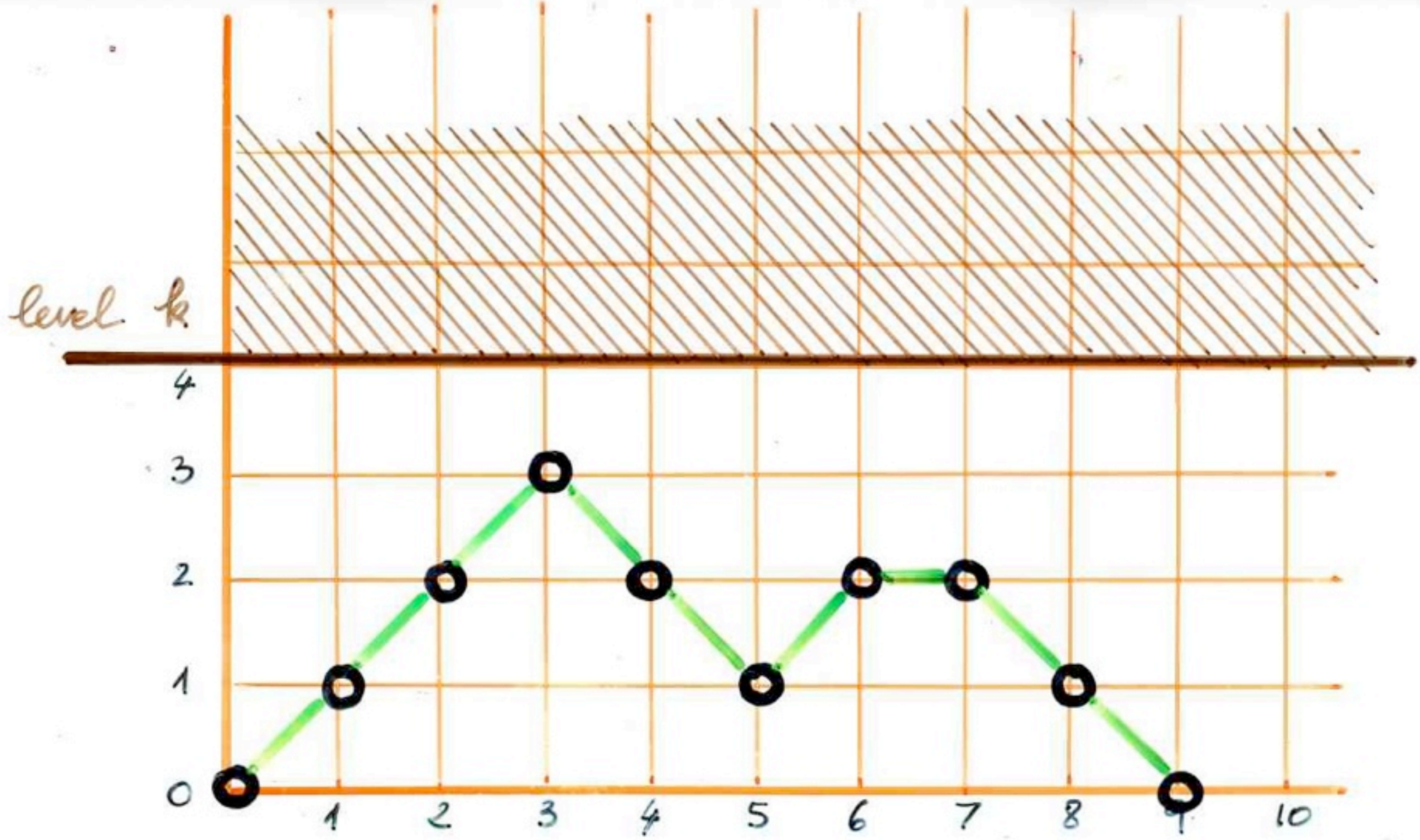
convergents

“réduites”

Jacobi continued fraction

$$J(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t -}}}}$$

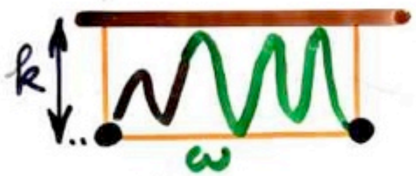
convergent $J_k(t)$ order k



level k
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10

convergents order k

Prop $$J_k(t) = \sum_{\substack{\omega \\ H(\omega)\le k}} v(\omega)$$



Prop $$J_k(t) = \frac{\delta P_k^*(t)}{P_{k+1}^*(t)}$$

Reciprocal

$$P_k^*(t) = t^k P_k\left(\frac{1}{t}\right)$$

- $P_k(x)$ orthogonal polynomials

defined by $\{(\lambda_k)_{k\geq 1} ; (b_k)_{k\geq 0}\}$

$$P_{k+1}(x) = (x - b_k)\,P_k(x) - \lambda_k\,P_{k-1}(x)$$

$$P_{-1}=0 \qquad P_0 = 1$$

- δP_k orthogonal polynomials

defined by $\{(\delta\lambda_k)_{k\geq 1} ; (\delta b_k)_{k\geq 0}\}$

$$\delta\lambda_k \downarrow \lambda_{k+1} \qquad \delta b_k \downarrow b_{k+1}$$

(formal) orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ \underbrace{f}_{\text{linear}} : \mathbb{K}[x] \longrightarrow \mathbb{K}$ linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n & (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 & \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 & \text{for } k \geq 0 \end{cases}$$

$$\mathcal{L}(x^n) = \mu_n \qquad (n \geqslant 0)$$

moments

$$\mathcal{L}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k\geqslant 1)$

3 terms linear recurrence relation

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$J(t) = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

$$P_{k+1}(x) = (x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

classical theory

continued fractions — **orthogonal polynomials**

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cdots\cdots\cfrac{}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

moments

generating function

$$P_{k+1}(x) = (x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path

$|\omega| = n$

example:
Hermite polynomials

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

moments
Hermite
polynomials

$$\cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}$$

atque series infinita ita se habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis est huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

moments

Hermite polynomials

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \dots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \dots, 2n\}$

chord diagrams

perfect matching

Hermite history

$$h = (\omega \; ; \; f)$$

ω: Dyck path

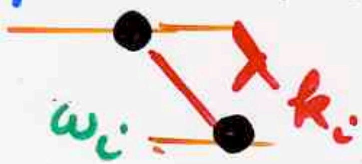
f: choice function

$\omega = \omega_1 \dots \omega_{2n}$

$f = (p_1, \dots, p_{2n})$

$p_i = 1$

$1 \leq p_i \leq v(\omega_i) = \lambda_{k_i}$



$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \dots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \dots, 2n\}$

chord diagrams

perfect matching

Hermite history

$$h = (\omega \ ; \ f)$$

ω: Dyck path

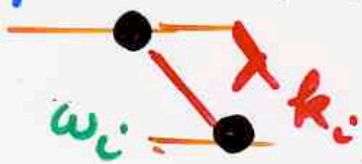
f: choice function

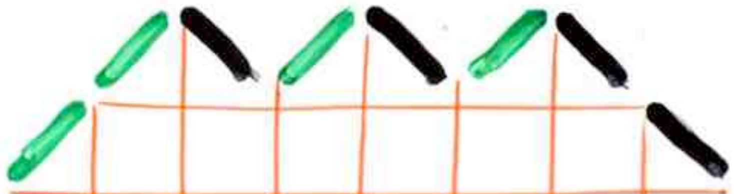
$$\omega = \omega_1 \dots \omega_{2n}$$

$$f = (p_1, \dots, p_{2n})$$

$$p_i = 1$$

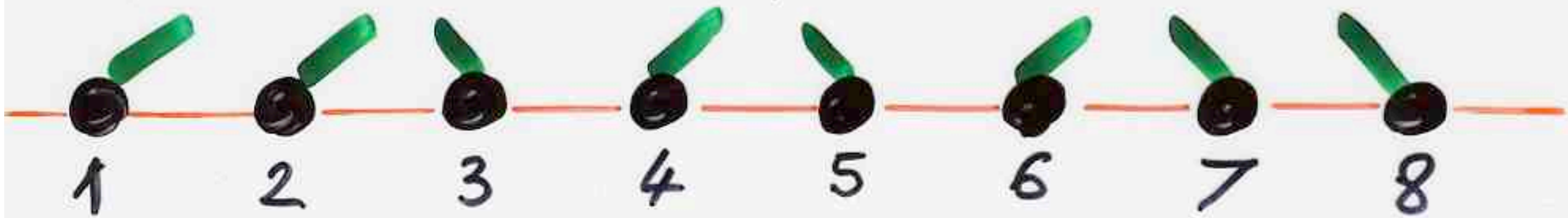
$$1 \le p_i \le v(\omega_i) = \lambda_{k_i}$$





	1	2	3	4	5	6	7	8
pos	1	1	1	1	2	1	1	1

"histories"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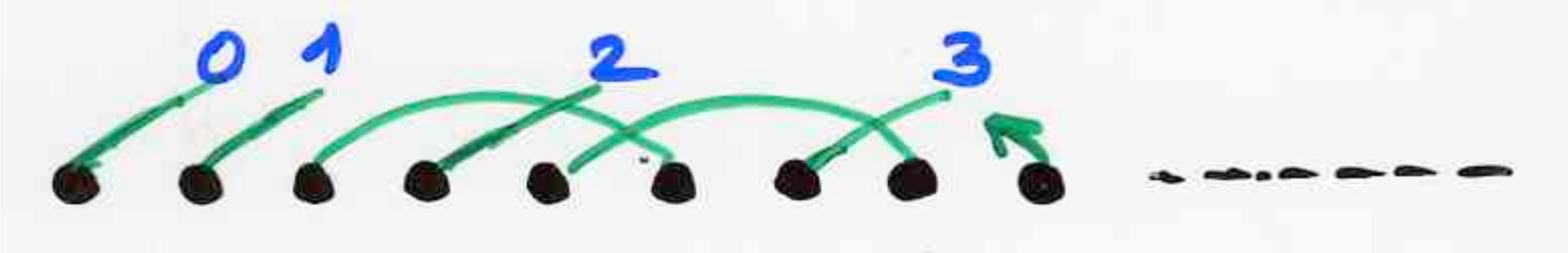
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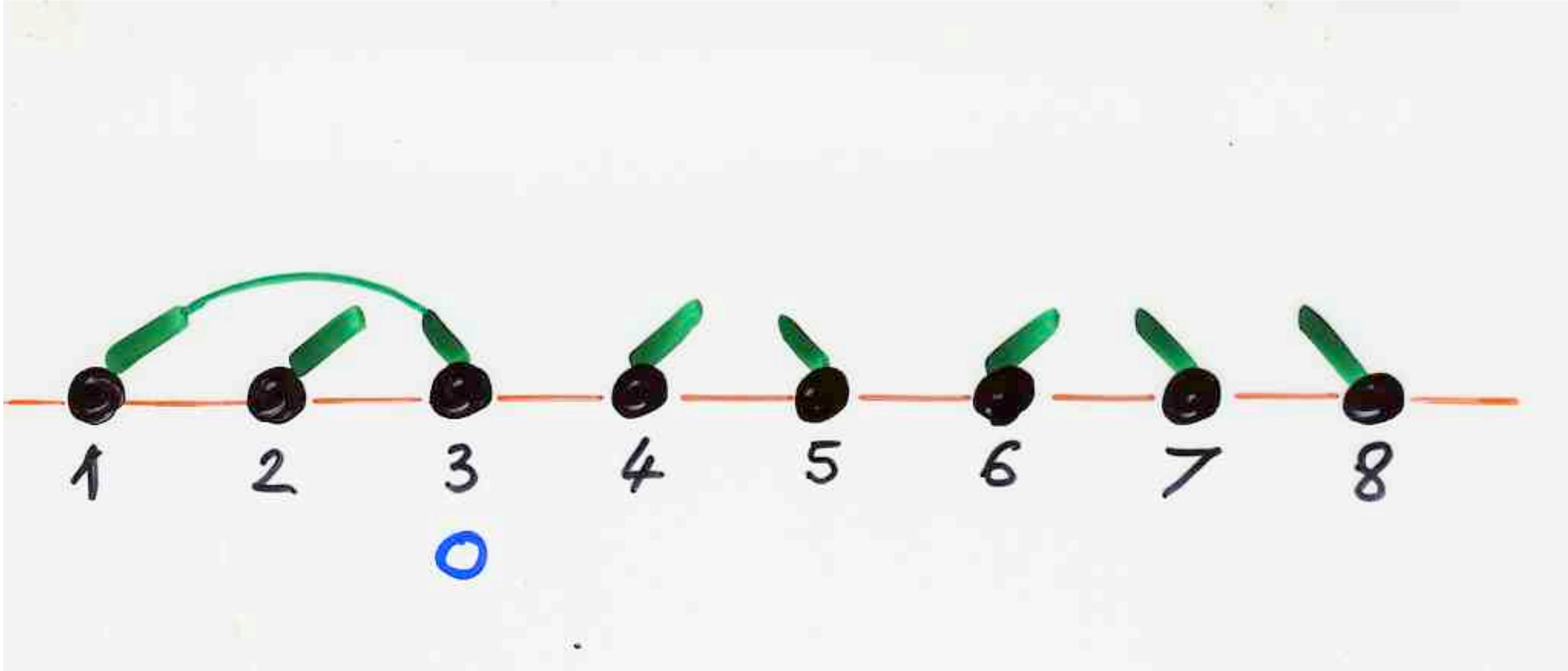
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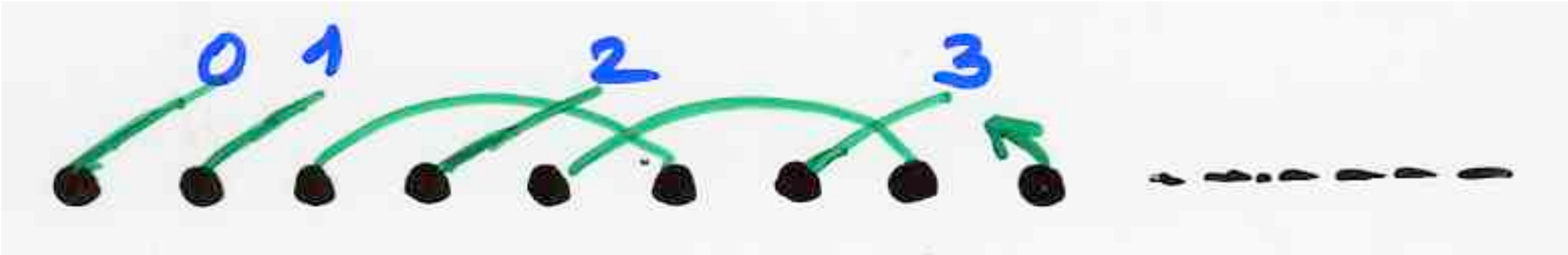
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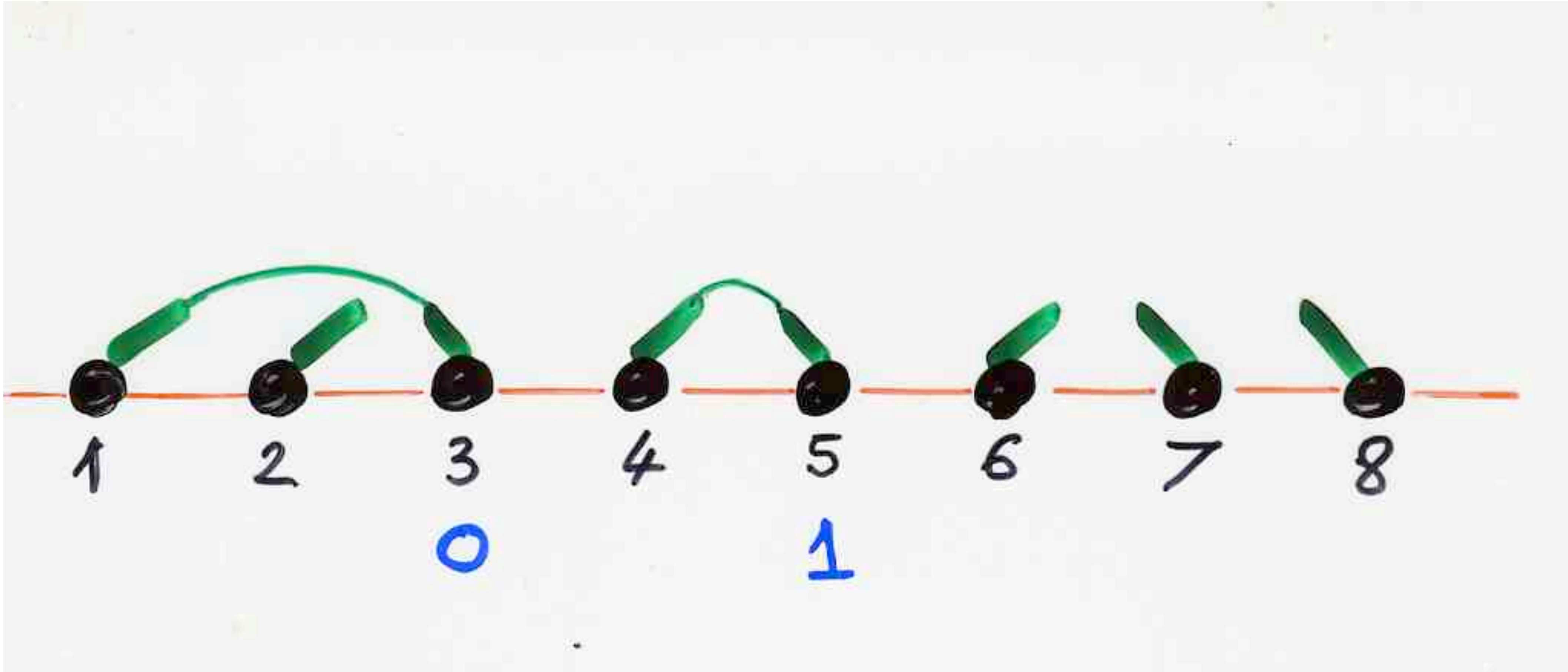
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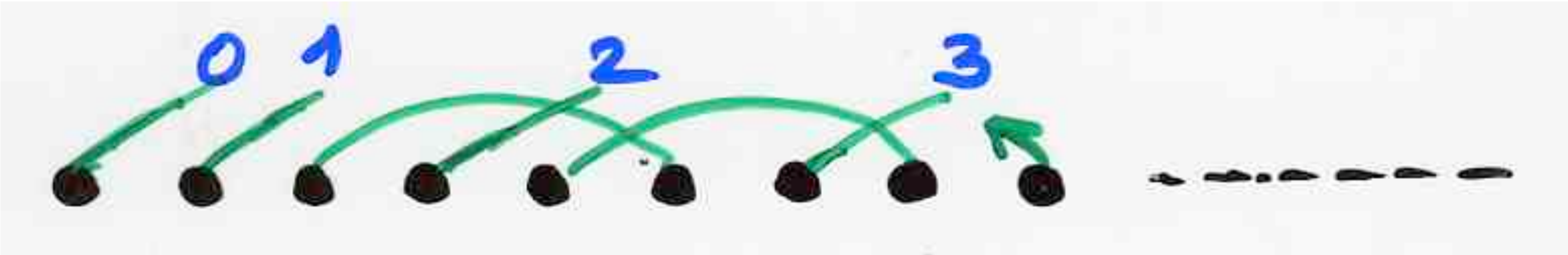
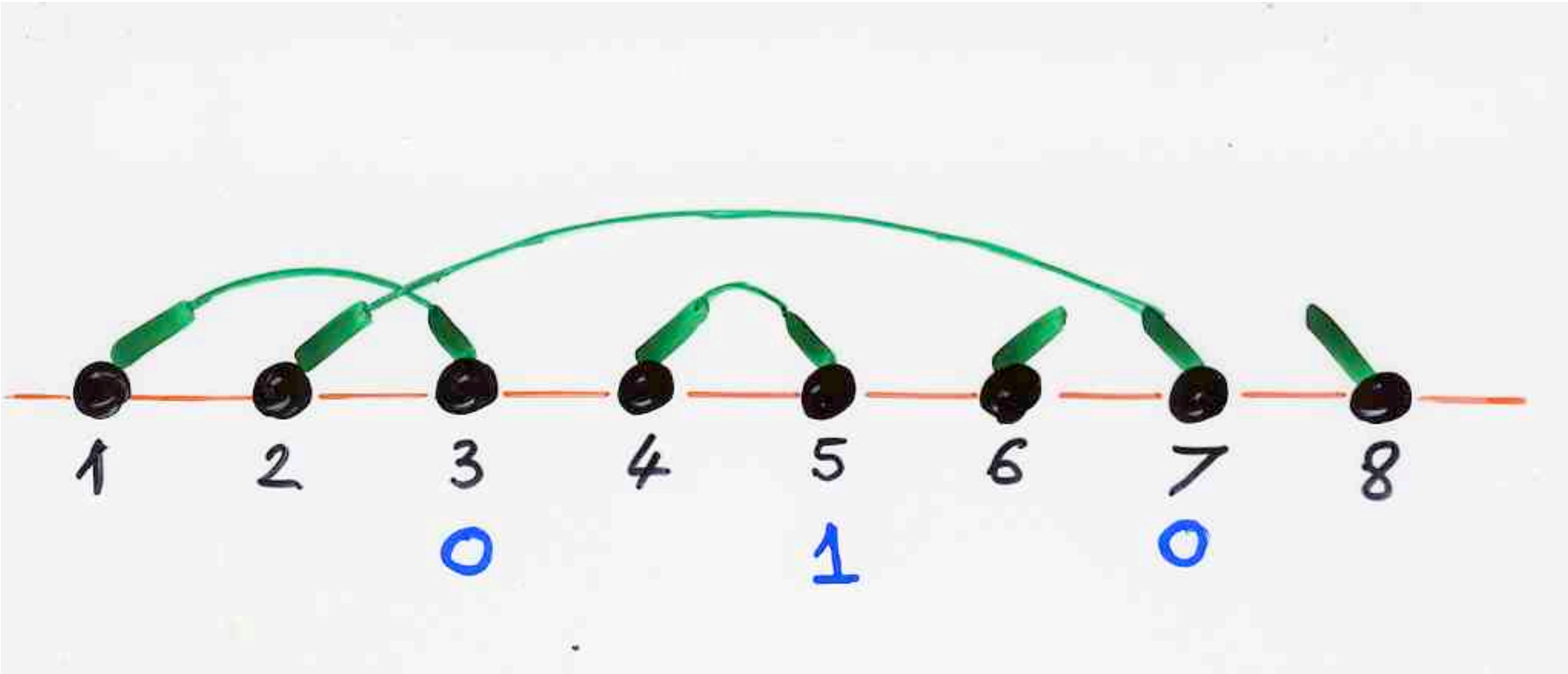

0 1 2 3


1 2 3 4 5 6 7 8
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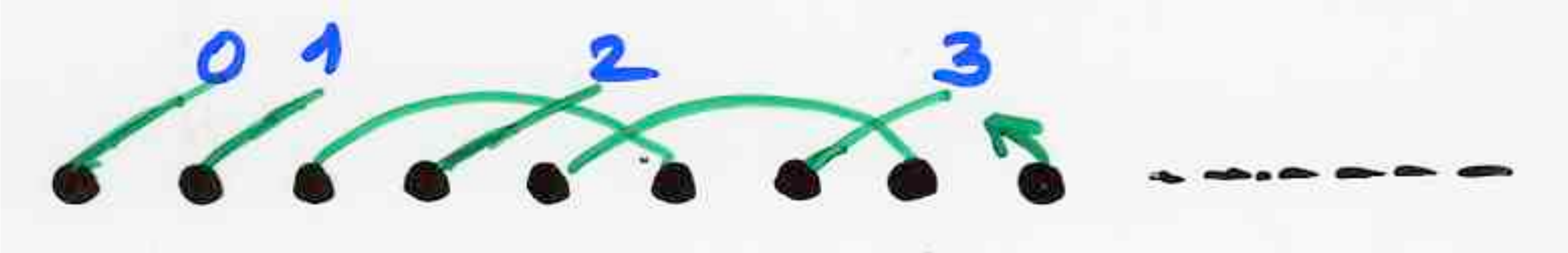
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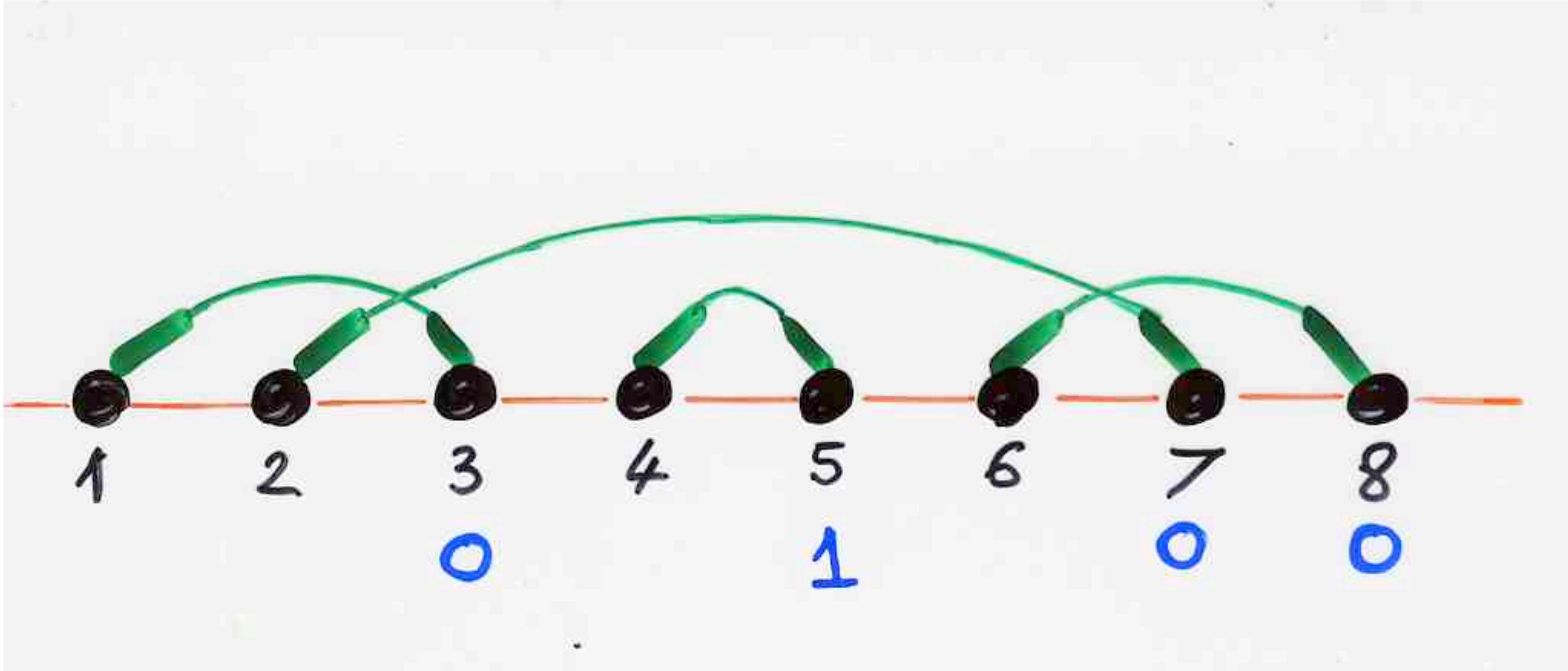

1 2 3 4 5 6 7 8
0 1 0 0

q-analog


www.mathinfo06.iecn.u-nancy.fr

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}}$$

$\lambda_k = q^k$

$t = -1$

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots}{1+\cfrac{q^k}{\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

q-analog of Hermite histories

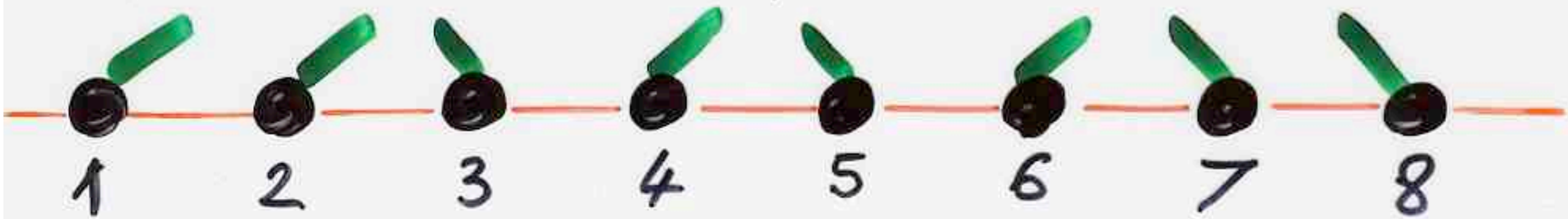
q-Hermite

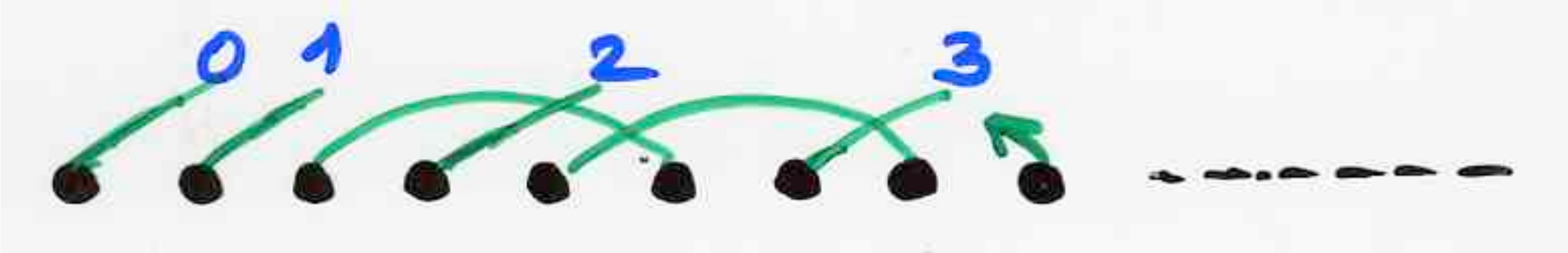
$H_n^{I}(x;q)$

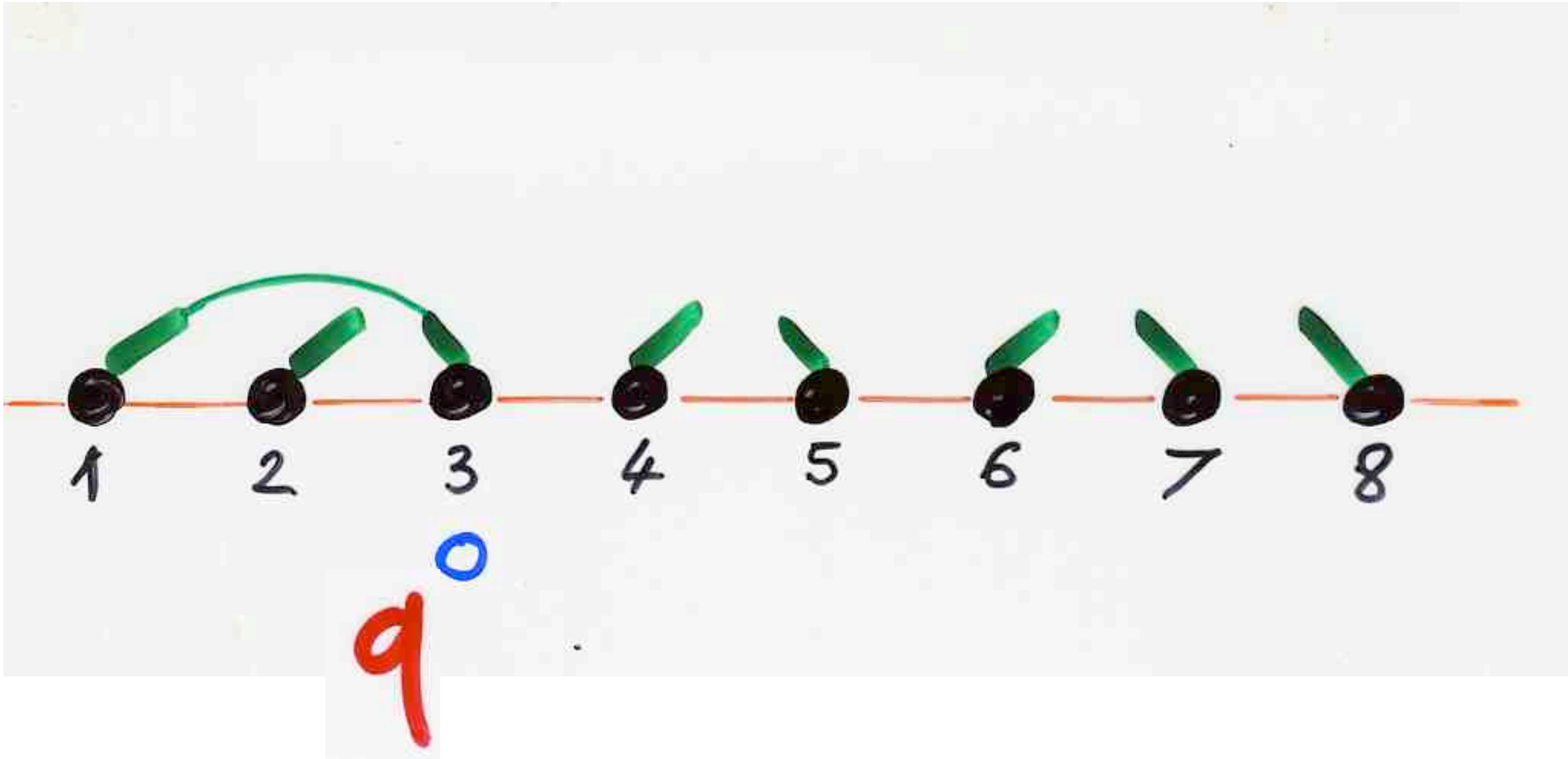
$b_k = 0$

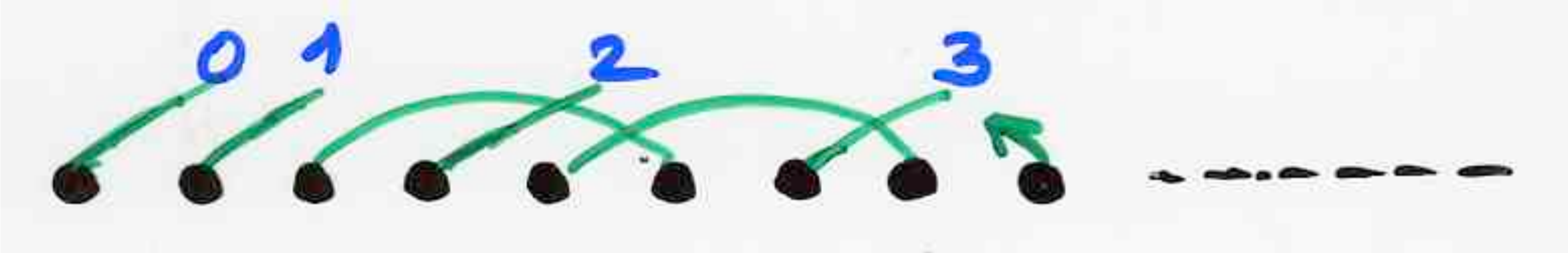
$\lambda_k = [k]_q$

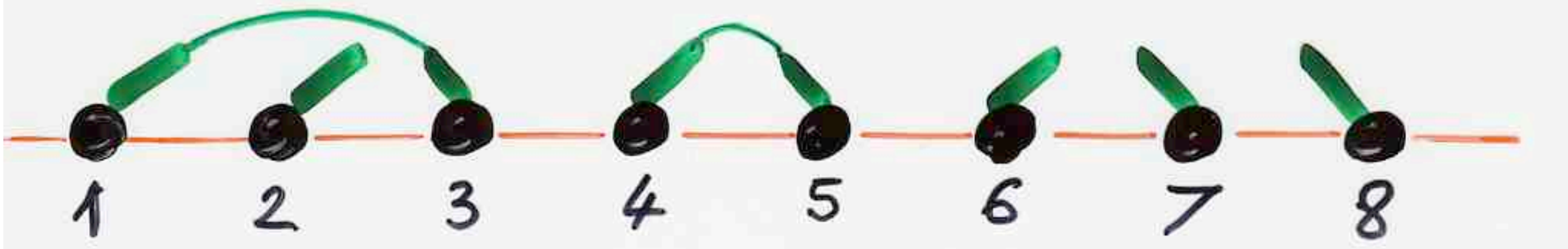
$= 1 + q + \dots + q^{k-1}$


1
2
3
4
5
6
7
8

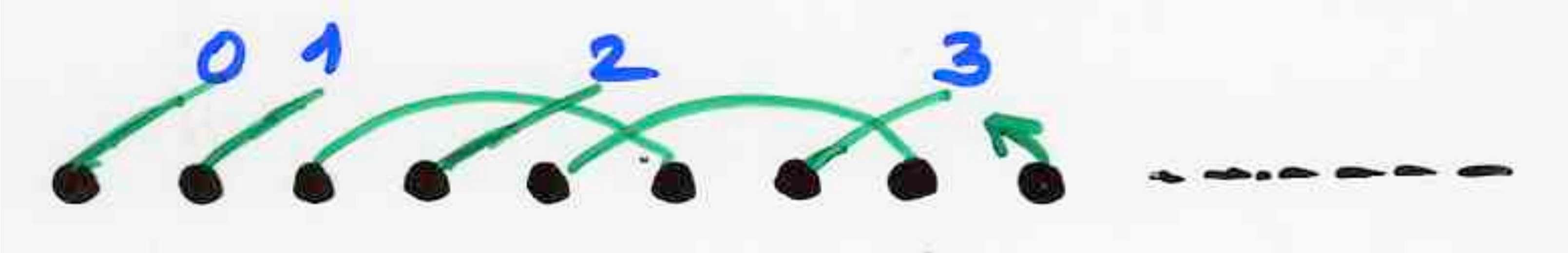

0
1
2
3

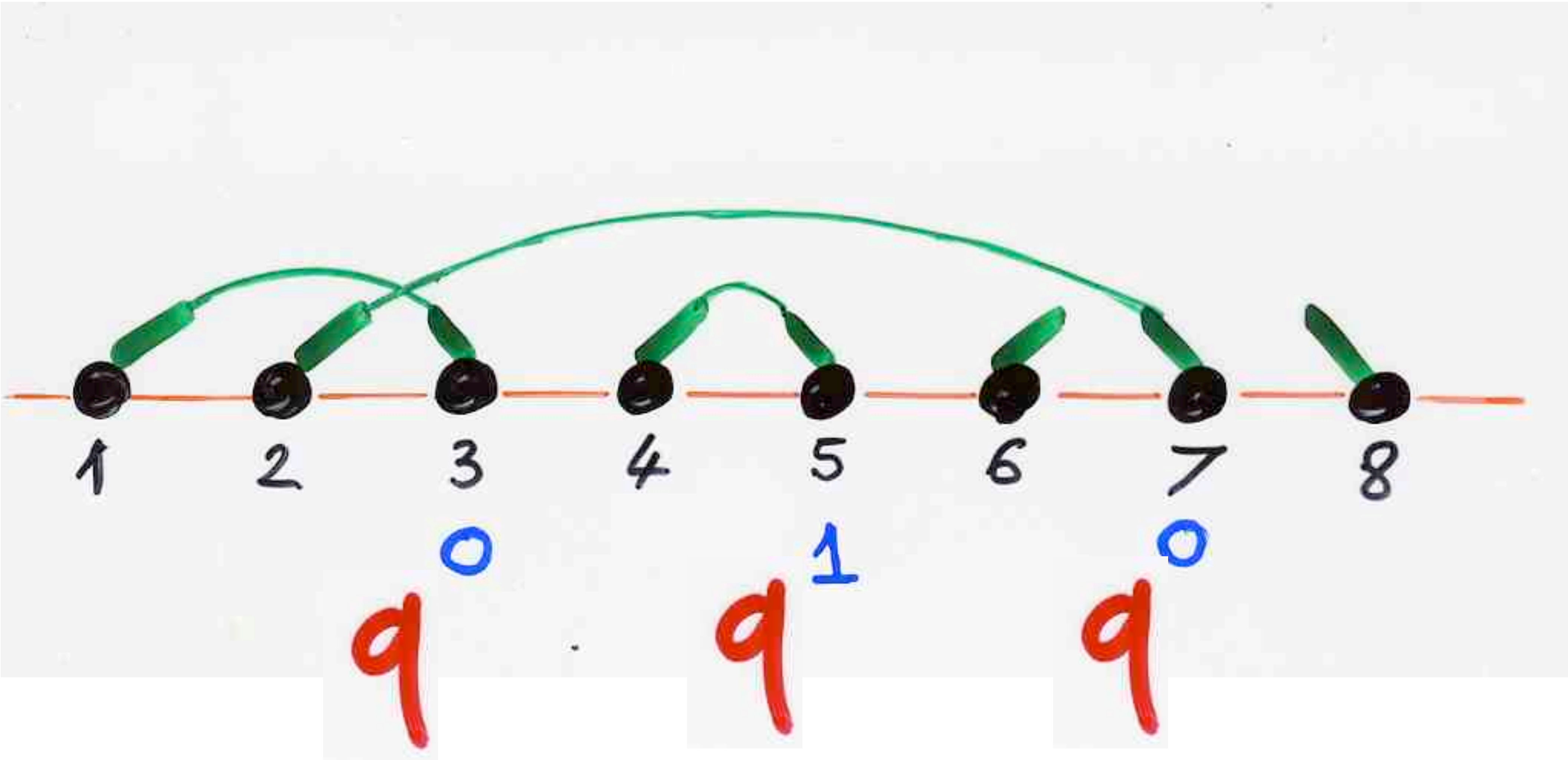

1
2
3
4
5
6
7
8
0
9



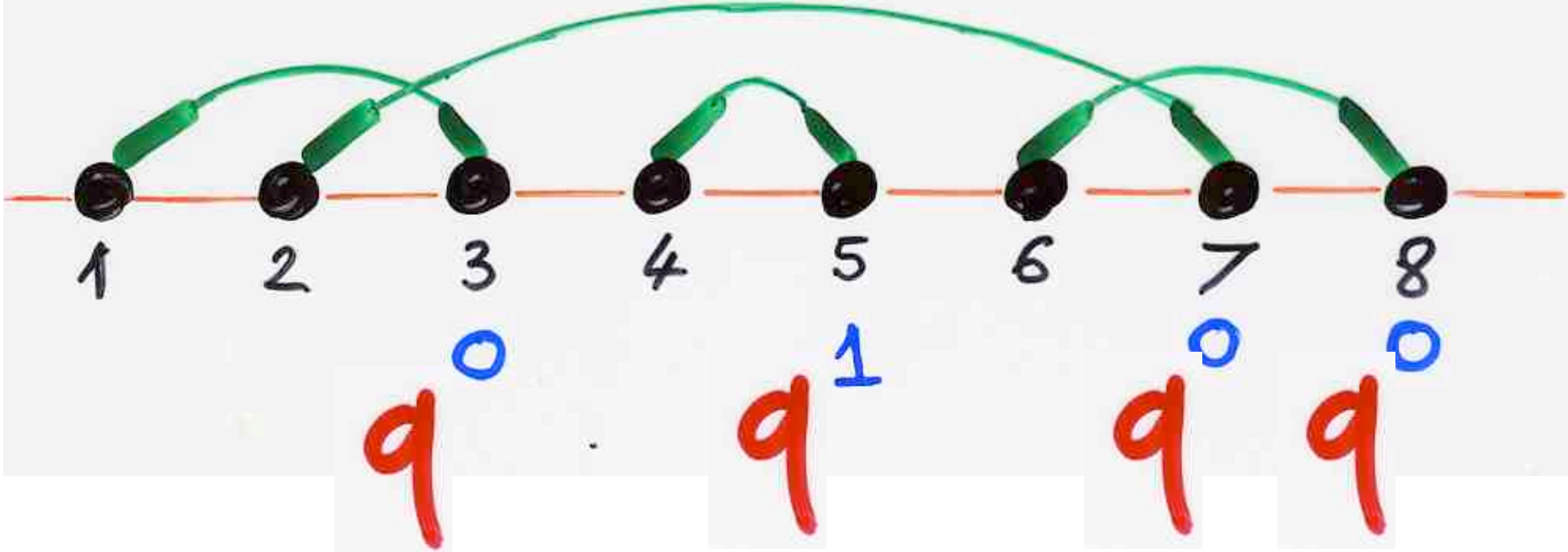


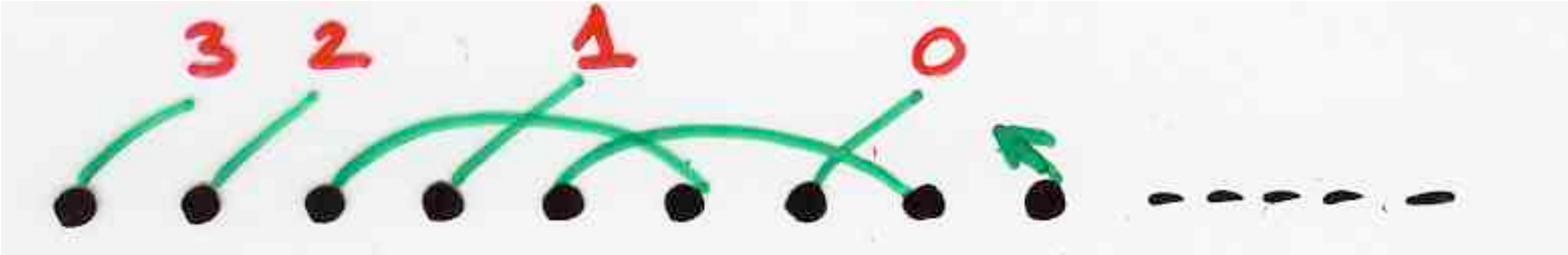
q^0 q^1


0 1 2 3

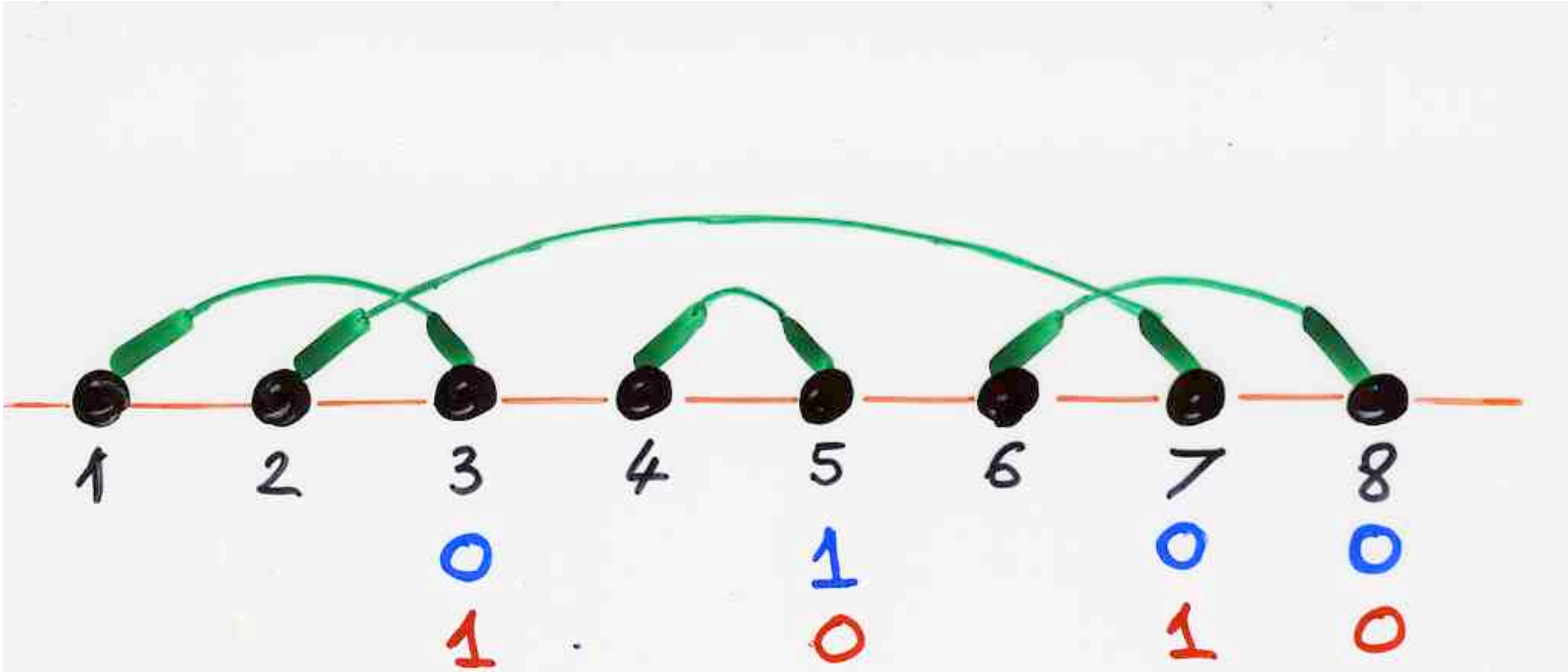

1 2 3 4 5 6 7 8
q^0 q^1 q^0

nesting

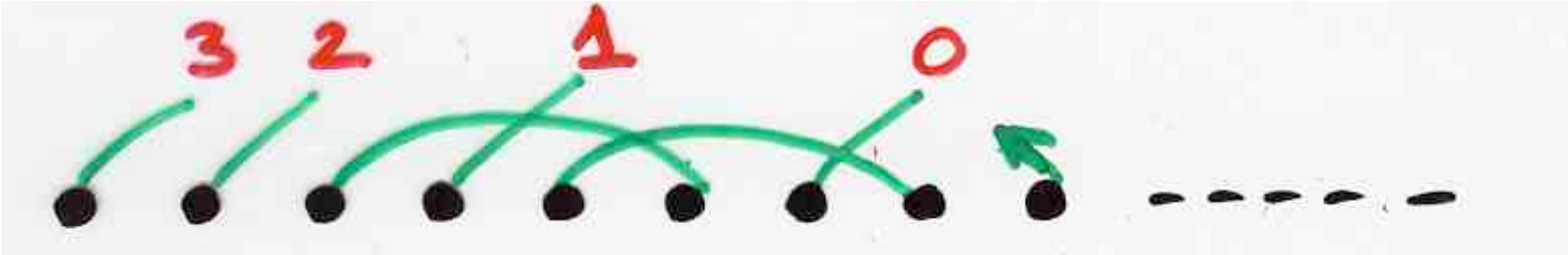




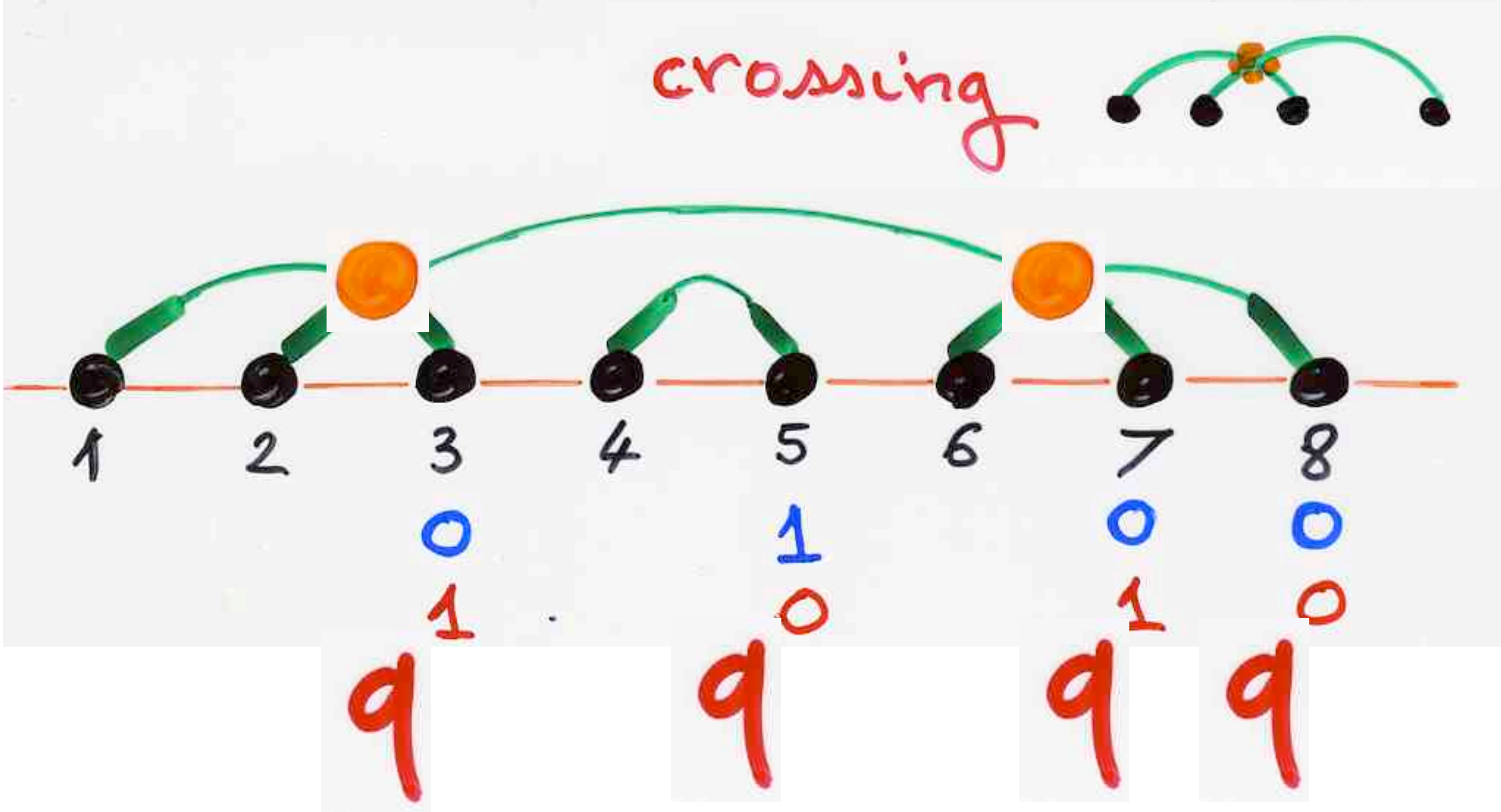
3 2 1 0



1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0



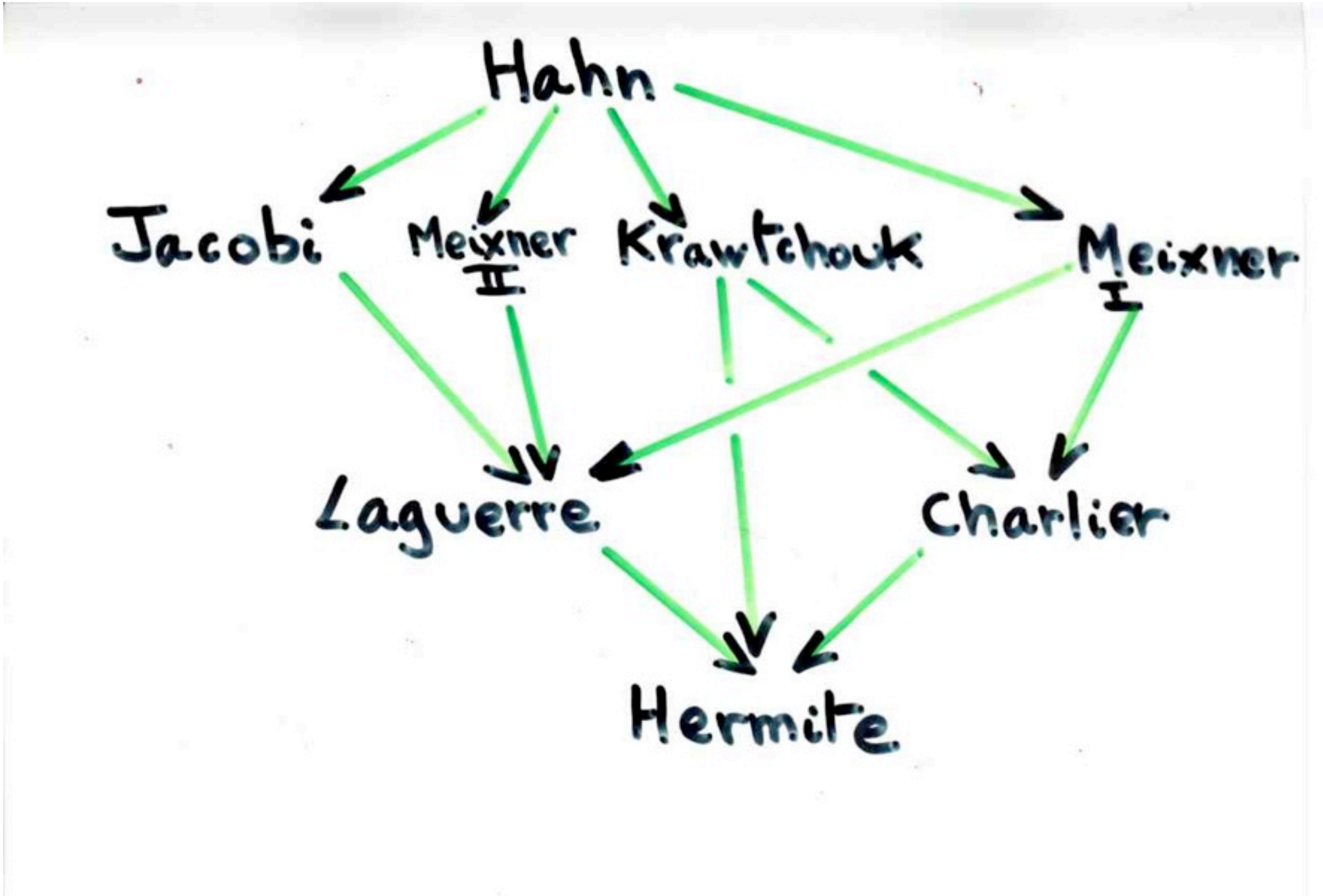
3 2 1 0



crossing
1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0
q q q q

Askey tableau

Askey-Wilson



Hermite $H_n(x) = (2x)^n \, {}_2F_0\left(\begin{matrix} -n/2, \frac{1-n}{2} \\ — \end{matrix}; \frac{-1}{x^2}\right)$

Laguerre $n! L_n^{(\alpha)}(x) = (\alpha+1)_n \, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right)$

Charlier $C_n^{(a)}(x) = {}_2F_0\left(\begin{matrix} -n, -x \\ — \end{matrix}; -\frac{1}{a}\right)$

Jacobi $n! P_n^{(\alpha,\beta)}(x) = (\alpha+1)_n \, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right)$

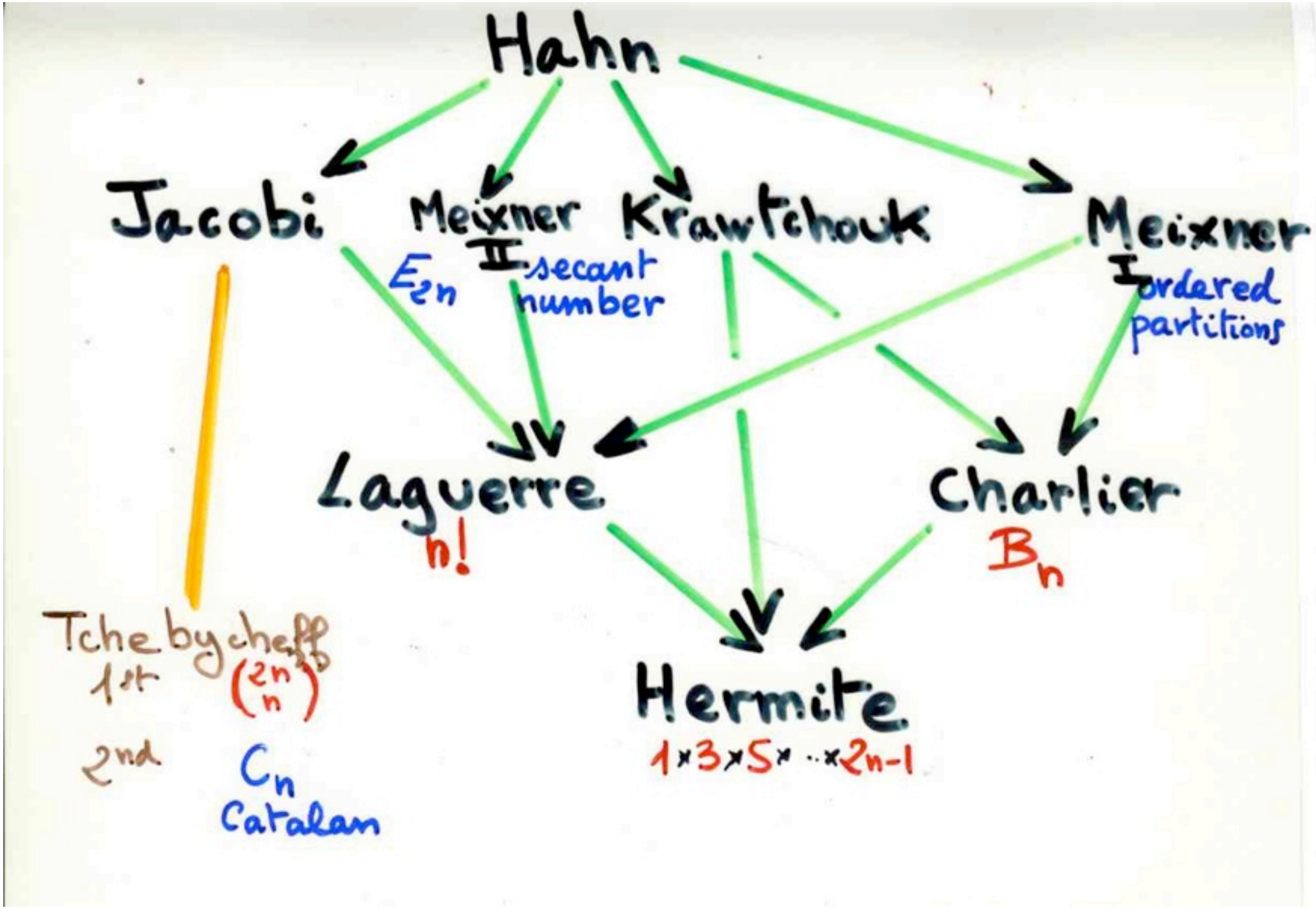
Meixner $m_n(x;\beta,c) = (\beta)_n \, {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1-c^{-1}\right)$

Krawtchouk $K_n(x;p,N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; p^{-1}\right)$

Meixner-Pollaczek $P_n^{a}(x;\varphi) = e^{in\varphi}\frac{(2a)_n}{n!} \, {}_2F_1\left(\begin{matrix} -n, a+ix \\ 2a \end{matrix}; 1-e^{-2i\varphi}\right)$

Hahn $Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right)$

Askey-Wilson



Laguerre histories

The FV bijection

Françon-XV 1978

Laguerre

polynomial

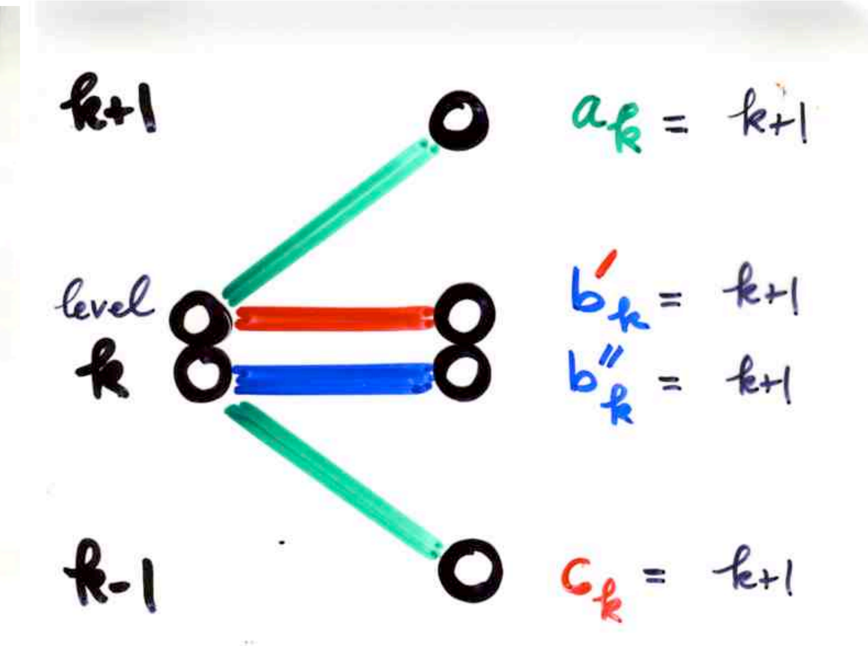
$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1 \qquad P_1 = x - b_0$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\cdots}}}$$



Permutations

n+1

(γ_c , f)

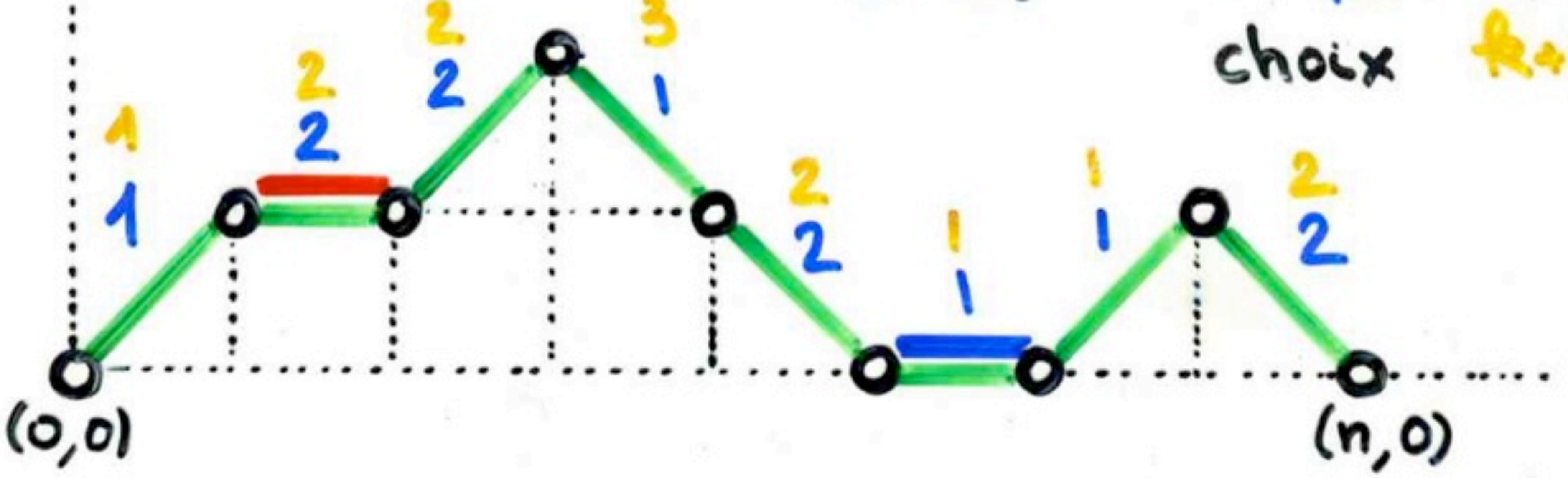
n

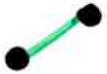
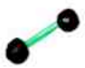
2 couleurs
paliers

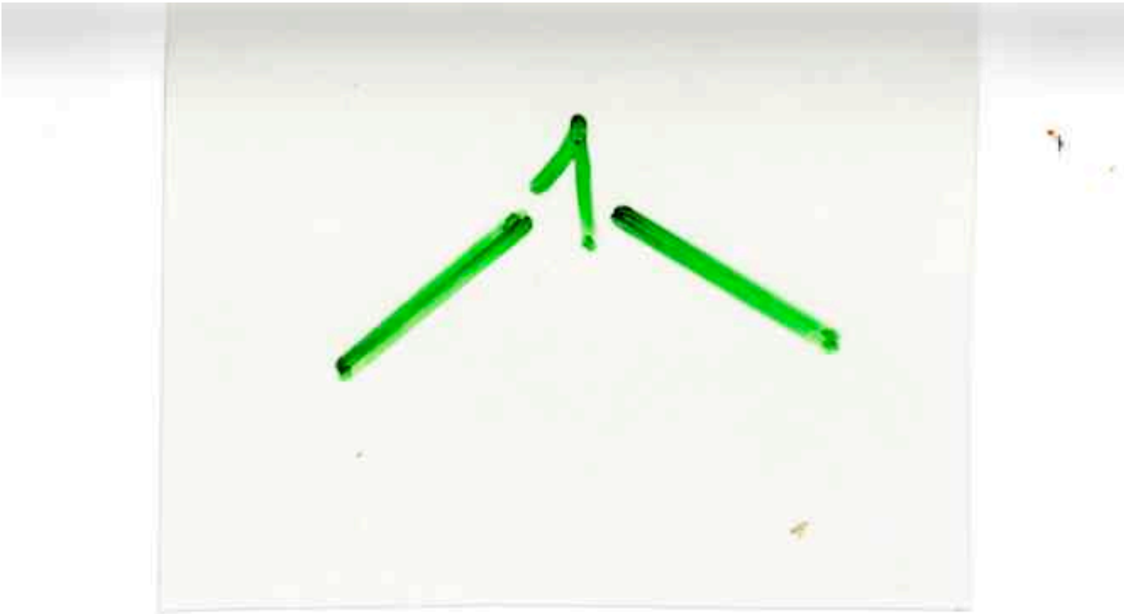
$f = (p_1, \dots, p_n)$

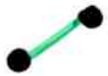
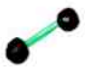
$1 \leq p_i \leq v(\omega_i)$

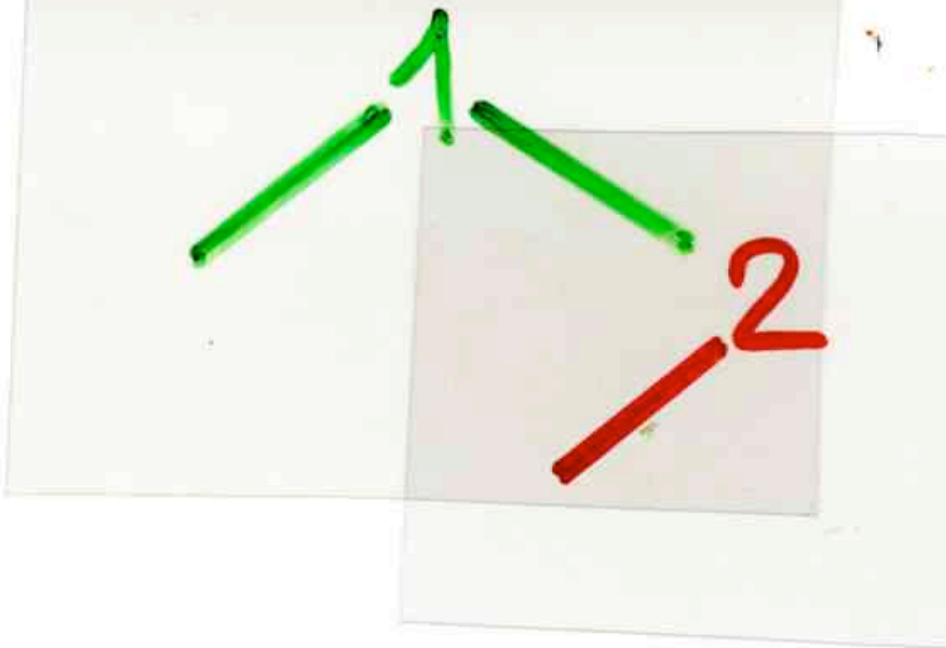
choix k+1



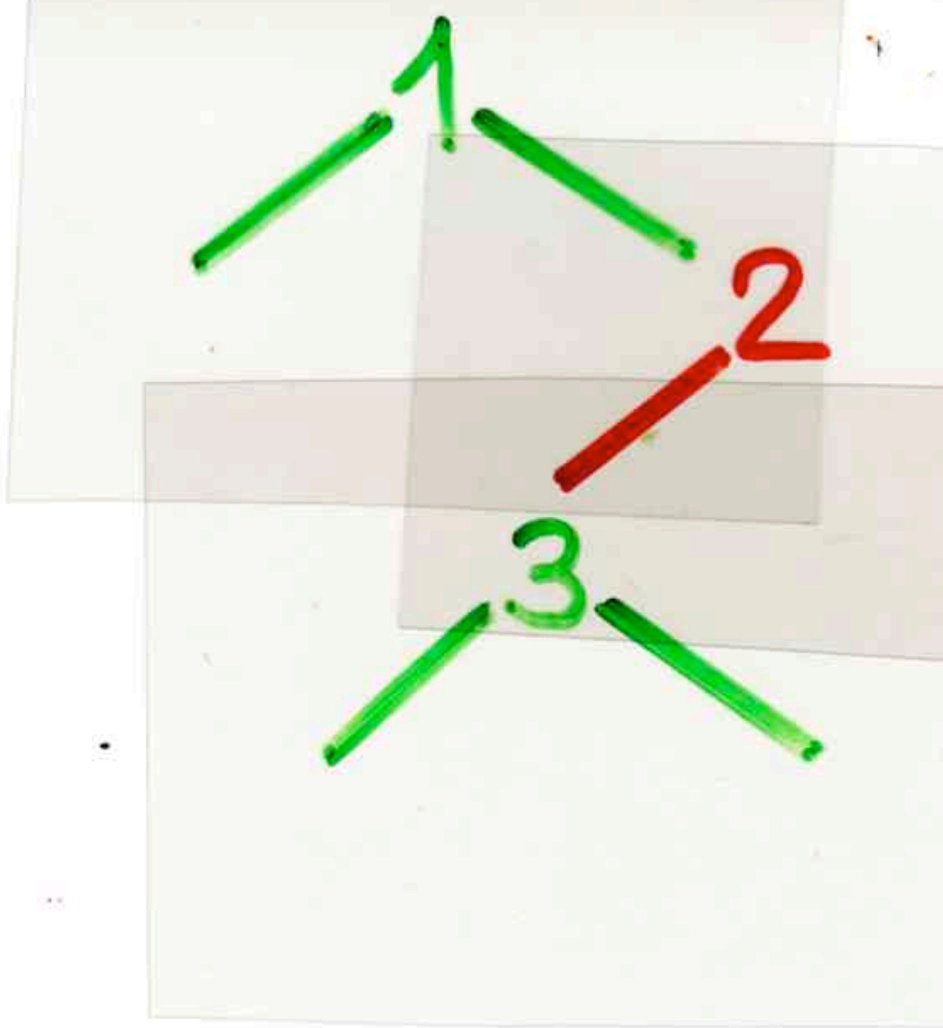
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



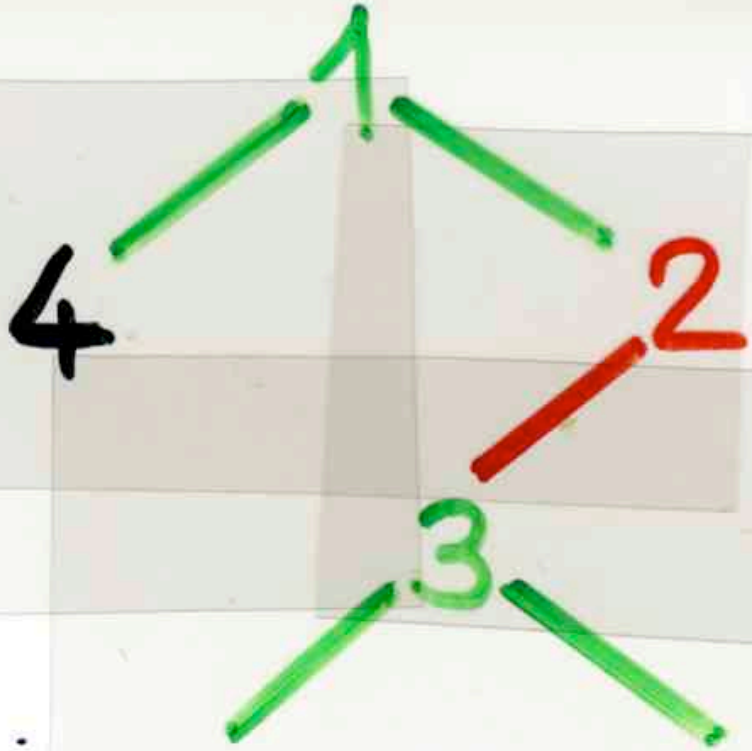
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



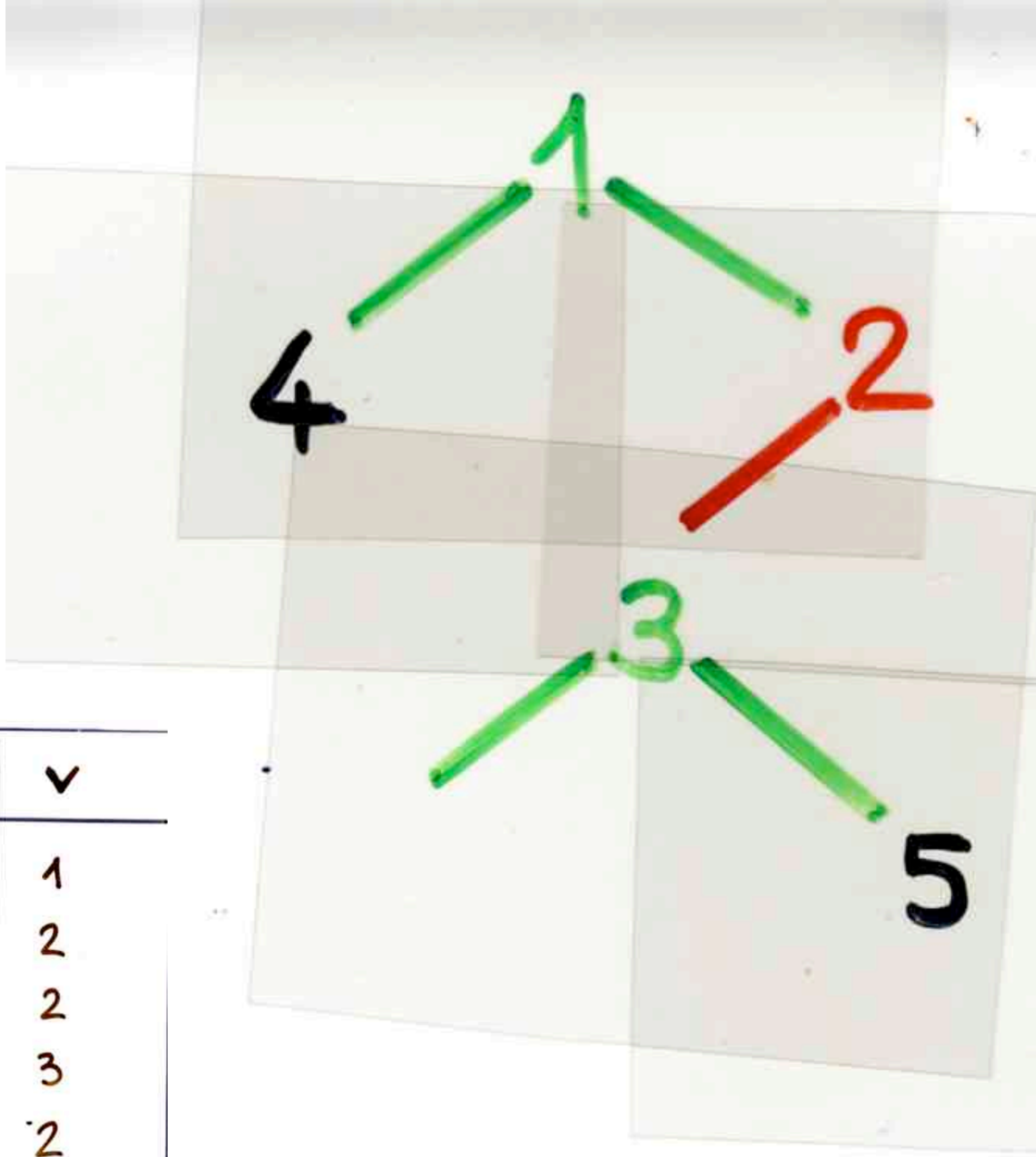
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



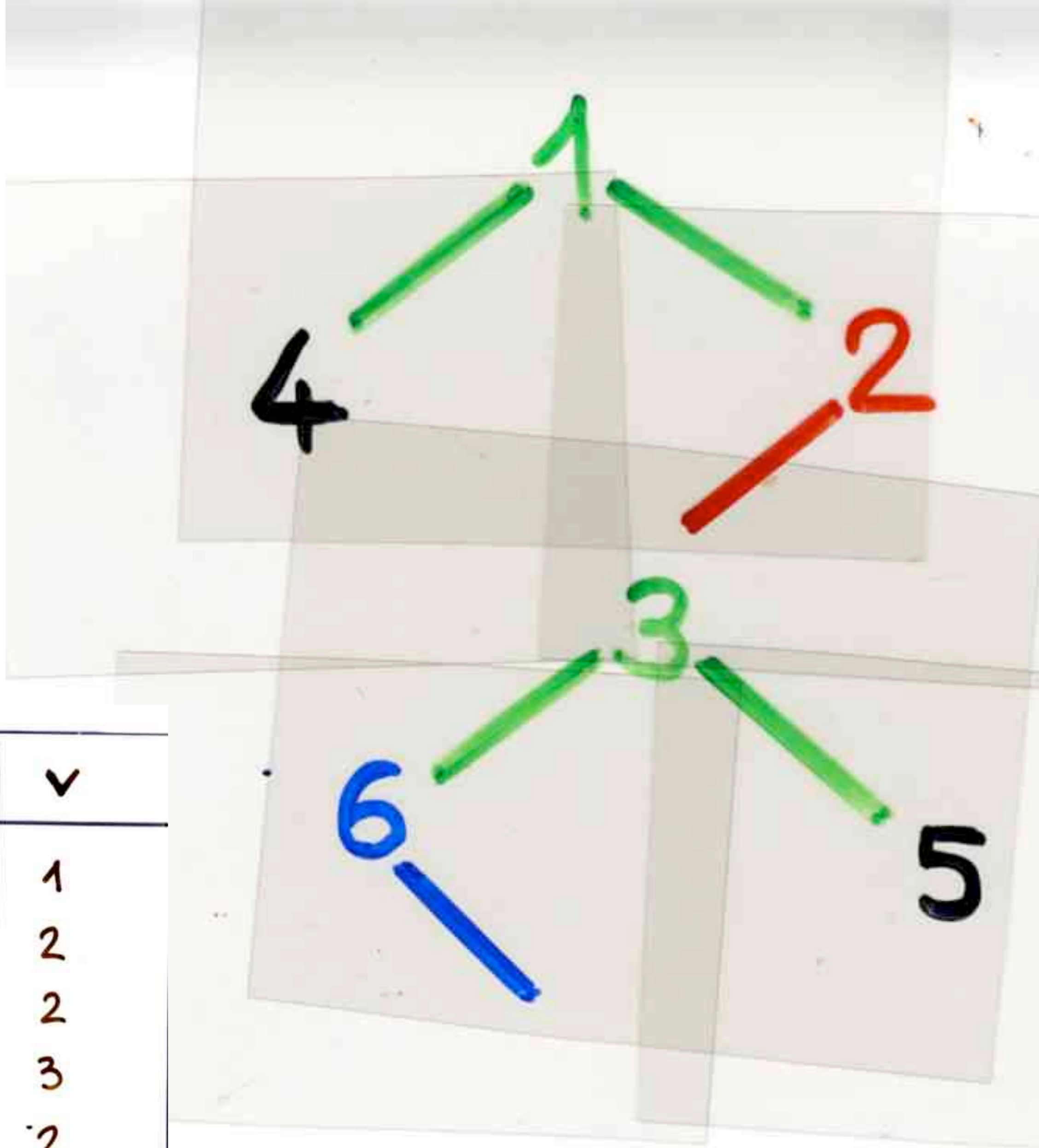
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			



x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

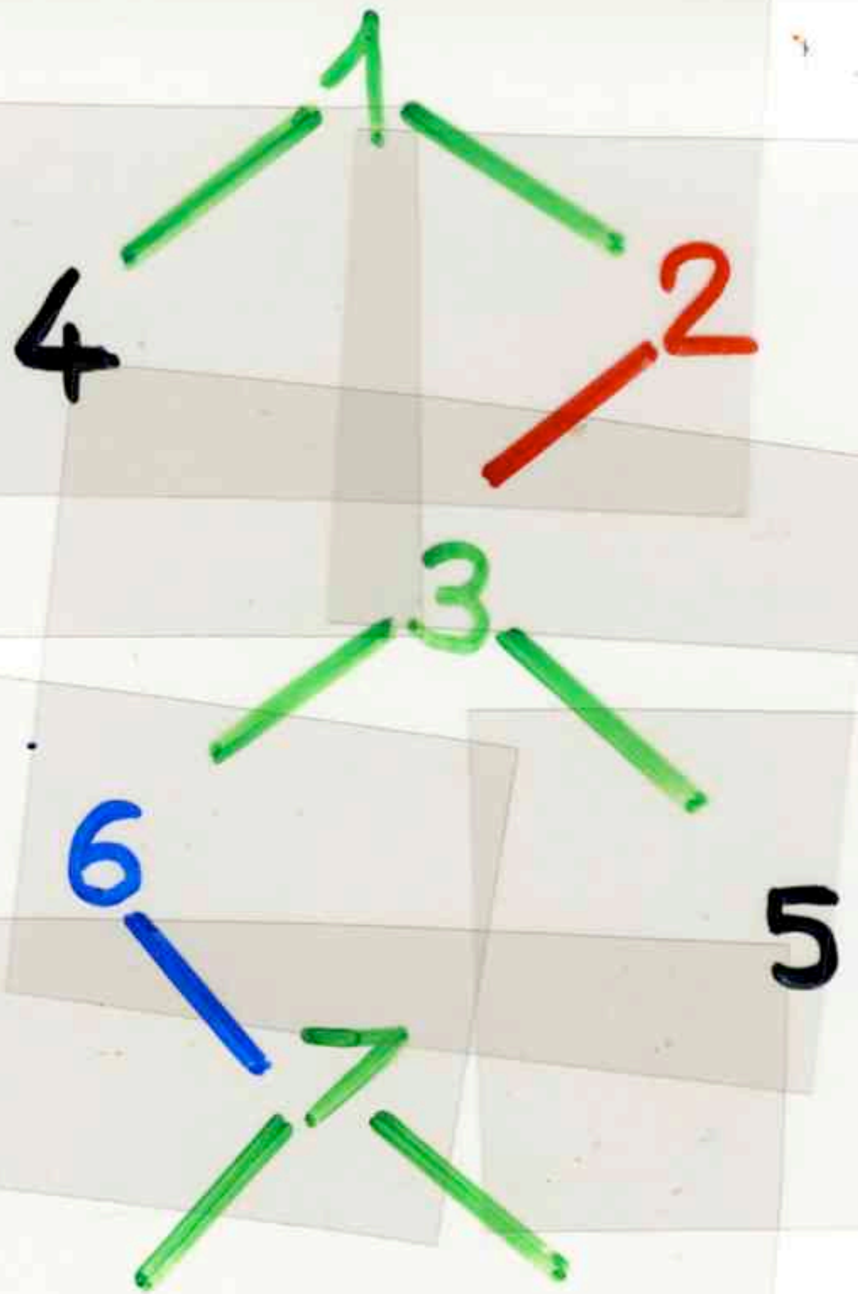


x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			

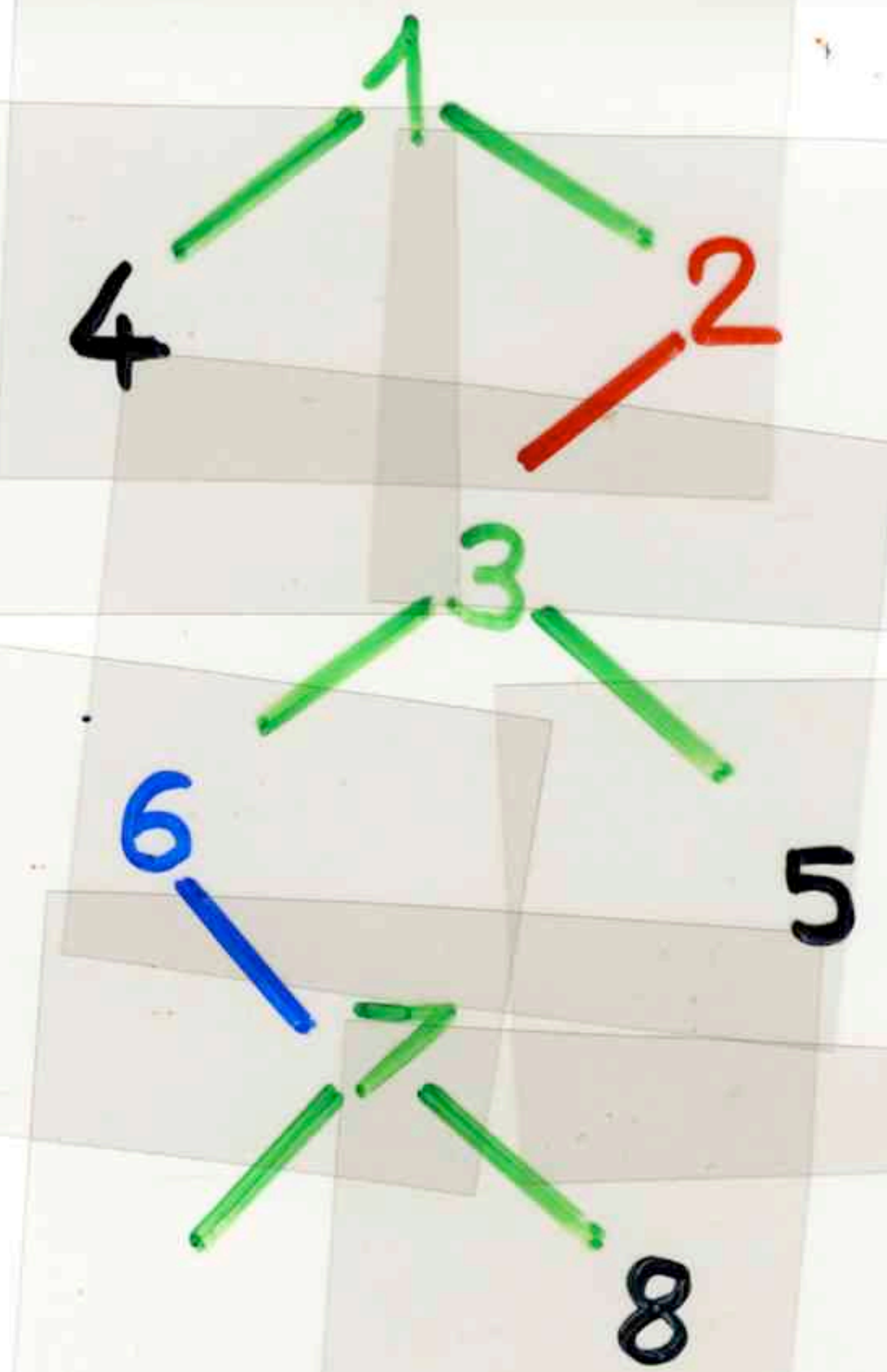


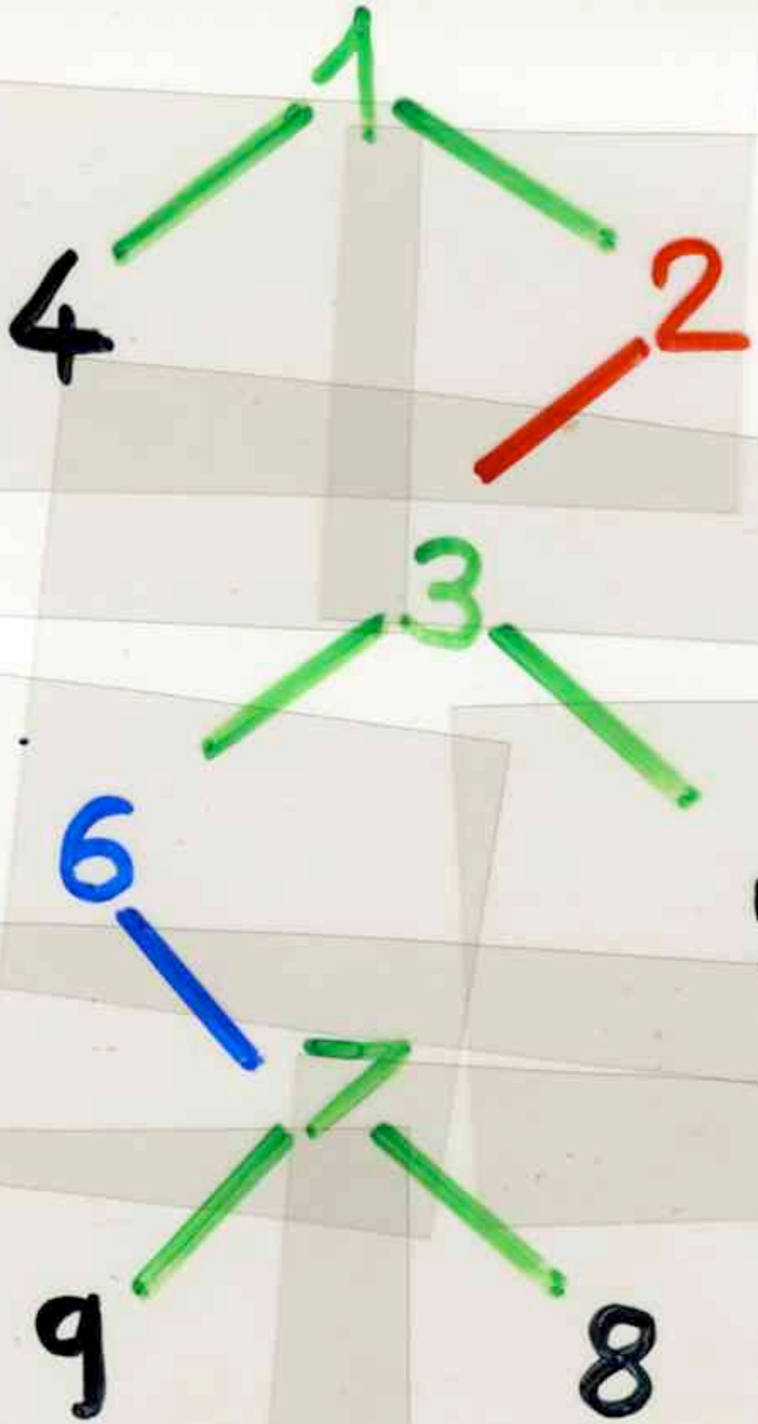
x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			

x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		



x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		





x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9	•		

4 1 6 9 7 8 3 5 2

$$\mathcal{L}_n \xrightarrow{\theta} \mathcal{T}_{n+1} \xrightarrow[\text{projection}]{\pi} \mathfrak{S}_{n+1}$$

histoires de Laguerre

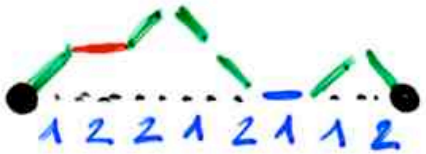
arbres binaires croissants

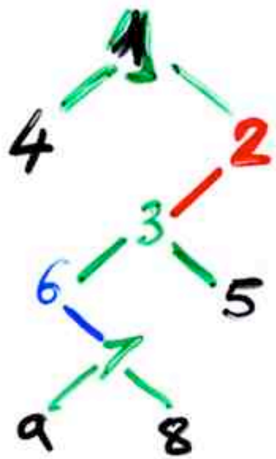
permutations

$$h = (\omega_c \,;\, (p_1, \dots, p_n))$$

chemin Motzkin coloré

fonction de possibilité





4 1 6 9 7 8 3 5 2

P. Biane

cycle structure

Foata, Zeilberger

de Médicis, X.V.

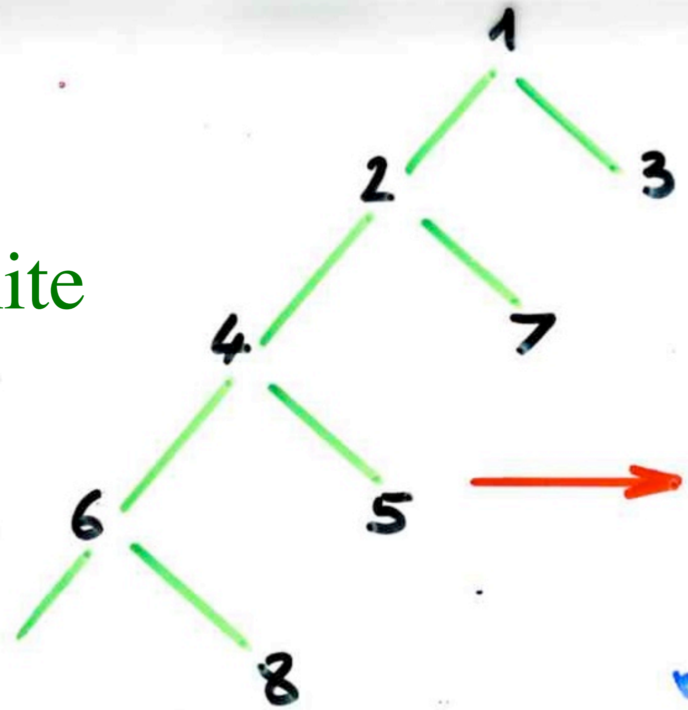
q-analog

Stieltjès

continued fraction

combinatorial proof for:
moments of orthogonal polynomials
or expansion in J-fraction

Hermite



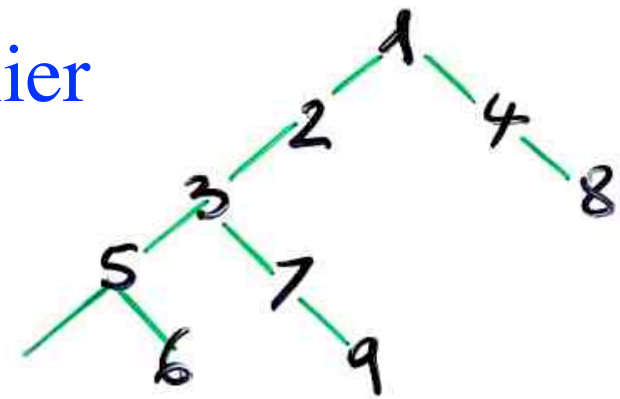
Involution

$\sigma = (1\,3)(2\,7)(4\,5)(6\,8)$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 5 & 4 & 8 & 2 & 6 \end{pmatrix}$

no fixed points

Charlier



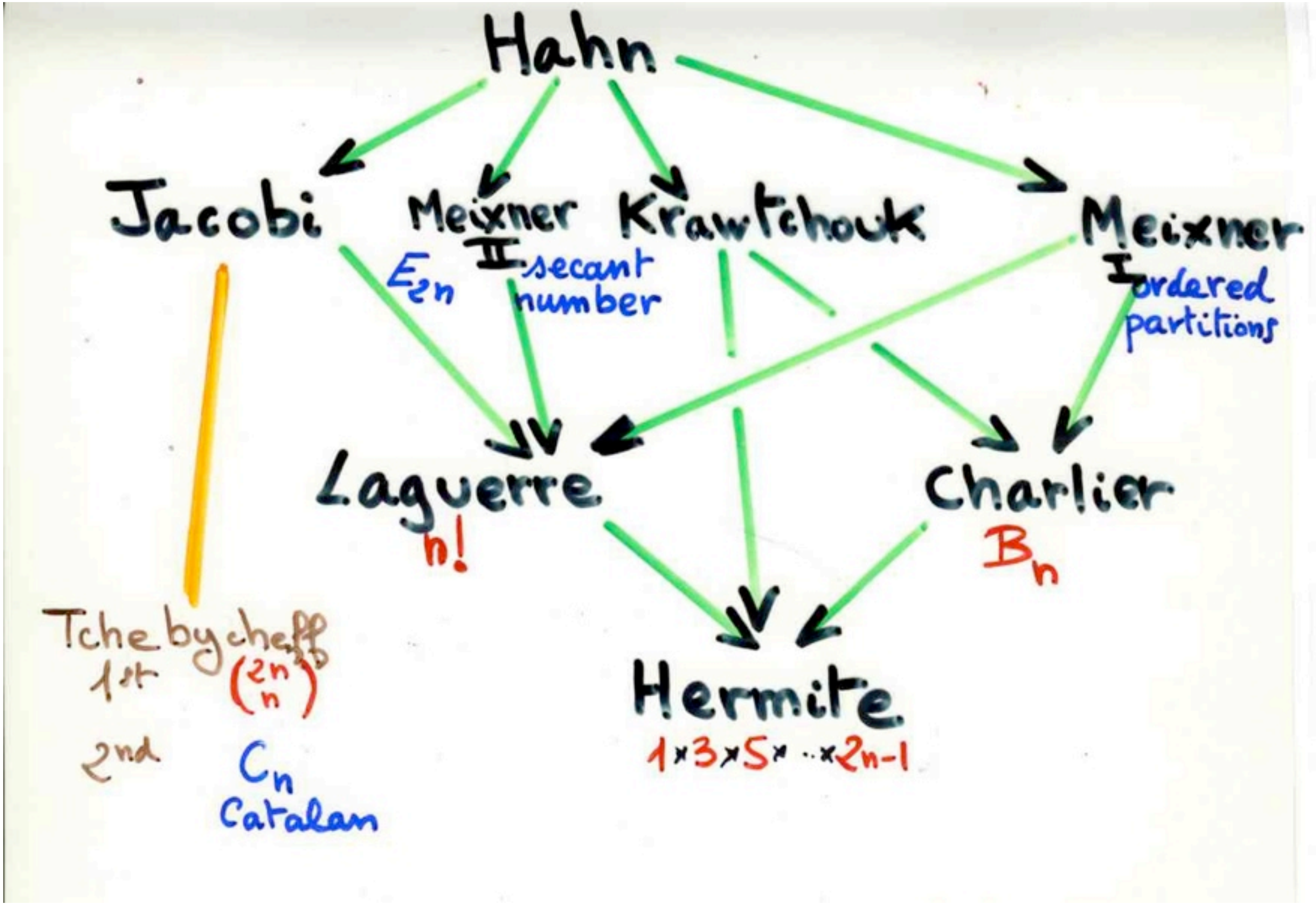
$\{1, 4, 8\}$

$\{2\}$

$\{3, 7, 9\}$

$\{5, 6\}$

Askey-Wilson



$$\mathrm{tg}(t) = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$\frac{1}{\cos(t)} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

E_{2n} $\{1, 5, 61, 1385, \dots\}$

nombres (d'Euler)
sécant

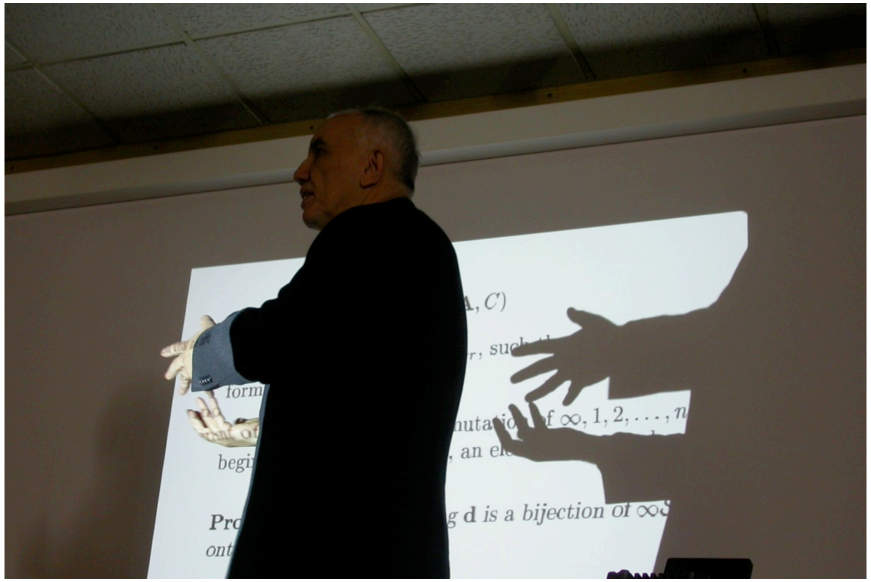
T_{2n+1} $\{1, 2, 16, 272, 7936, \dots\}$

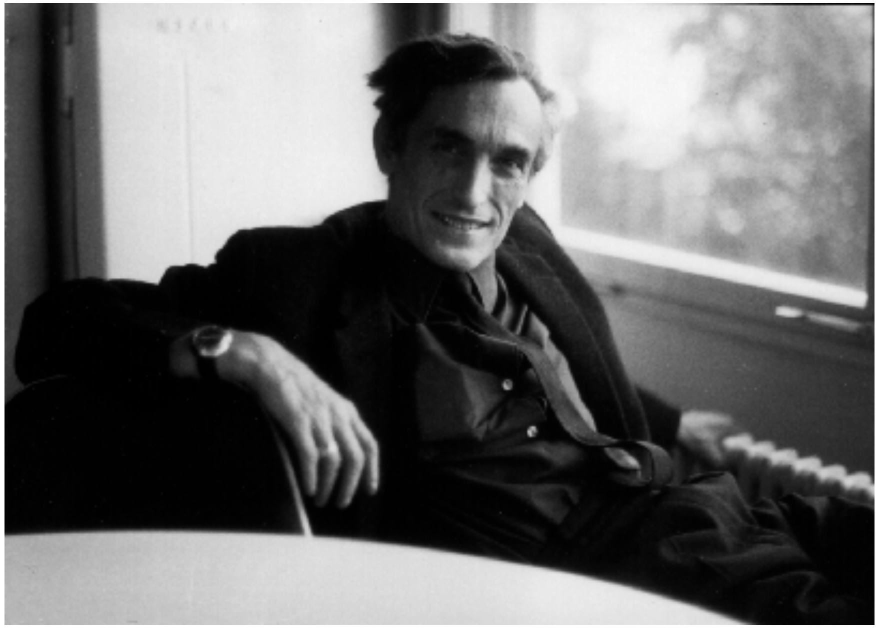
nombres
tangents

Permutations alternantes

D. André (1880)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$





D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens"
(1970)

$$\int_0^\infty e^{-t}\tan(tu)\,dt = \cfrac{1}{1-\cfrac{1.2\,u^2}{1-\cfrac{2.3\,u^2}{1-\cfrac{3.4\,u^2}{\cfrac{\cdots}{1-k(k+1)u^2\over \cdots}}}}}$$

Laplace transform

orthogonal

polynomials

(binomial type)

Scheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

orthogonal
polynomials

(binomial type)
Scheffer type

$$\sum P_n(x) \frac{t^n}{n!} = g(t) e^{x f(t)}$$

- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1}\, c_k$	Moments
Tchebycheff unitaires $U_n(x)$	0	$1/4$	$\frac{1}{4^n} C_n$ Catalan
$T_n(x)$	0	$1/4$ $\lambda_0 = 1/2$	$\frac{1}{4^n}\binom{2n}{n}$
Laguerre $L_n^1(x)$	$2k+2$	$k(k+1)$	$(n+1)!$
$L_n^{\alpha}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)\dots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1.3\dots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{a}(x)$	$k+a$	ak	$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x;\beta,c)$	$\frac{(1+c)k+\beta c}{1-c}$	$\frac{c\,k(k-1+\beta)}{(1-c)^2}$	$\frac{\sum_{\sigma\in\mathfrak{S}_n} \beta^{a(\sigma)} c^{1+d(\sigma)}}{(1-c)^n}$
Kreweras $\beta = 1$ $c = 1/2$	$3k+1$	$2k^2$	$= (1-c)^{\beta} \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$
Meixner II $M_n(x;\delta,\eta)$	$(2k+\eta)\,\delta$	$(\delta^2+1)\,k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{a(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$
$\delta = 0$ $\eta = 1$	0	k^2	E_{2n} Sécant

enumerative
algebraic combinatorics
bijective

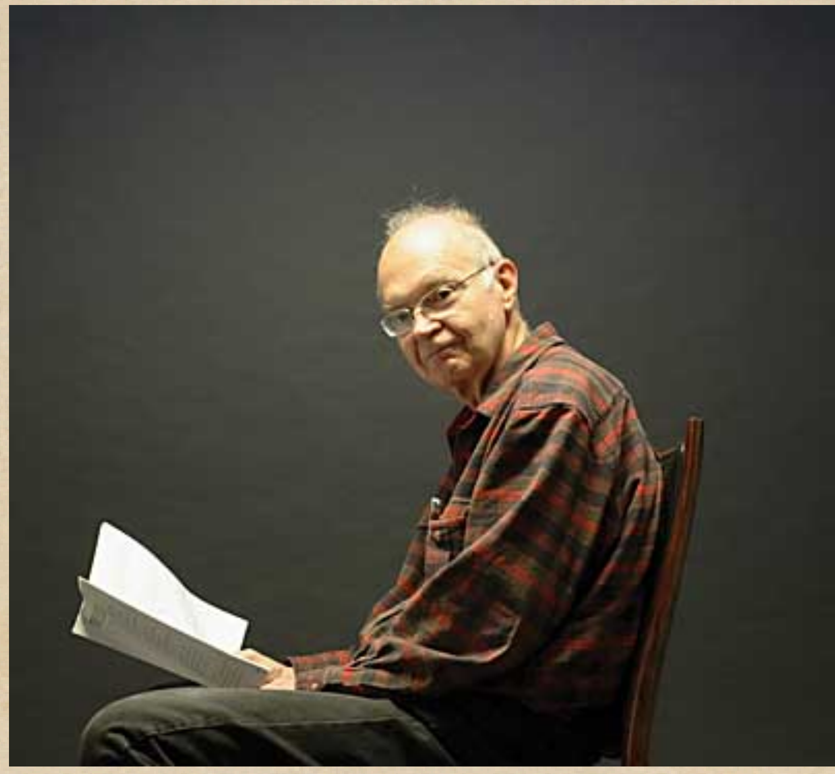
analytic combinatorics

FFVV 1976 - 79


www.mathinfo06.iecn.u-nancy.fr

data structure

integrated cost



Calcul du coût intégré

... d'une structure de données

pour une séquence aléatoire

d'opérations primitives

Françon, Flajolet, Vuillemin (1980, ...)

connaissant le coût moyen

d'une opération primitive.

J. Françon 1976

data structure histories

"histoires de fichiers"

24

17

10

8

24

17

12

10

8

Opérations primitives

A ajout

S suppression

I_+ positive

interrogation

I_- négative

Opérations primitives

A ajout

S suppression

I_+ interrogation positive

I_- interrogation négative

Laguerre $L_n^{(1)}(x)$

$\mu_n = (n+1)!$

$a_k = k+1$

$b'_k = k+1$

$b''_k = k+1$

$c_k = k+1$

Laguerre $L_n^{(1)}(x)$

$\mu_n = (n+1)!$

$\mu_n = n!$ $L_n^{(0)}(x)$

$a_k = k+1$ $k+1$

$b'_k = k+1$ k

$b''_k = k+1$ $k+1$

$c_k = k+1$ k

("abstract") Data structures	Possibility functions a_k	q_k	s_k	Number of Histories. h_n
Dictionary	$k+1$	$2k+1$	k	$n!$ Permutations
Linear list	$k+1$	0	k	E_{2n} alternating permutations
Priority Queue	$k+1$	0	1	$1.3 \ldots (2n-1)$ involutions with no fixed pts.
Symbol table	$k+1$	k	1	$B_n^{(2)}$ Partitions
Stack	1	0	1	$C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan nb.

("abstract") Data structures	Possibility functions a_k	q_k	s_k	Number of Histories. Moments h_n	Orthogonal Polynomials
Dictionary	$k+1$	$2k+1$	k	$n!$ Permutations	Laguerre
Linear list	$k+1$	0	k	E_{2n} alternating permutations	Meixner
Priority Queue	$k+1$	0	1	$1.3\dots(2n-1)$ involutions with no fixed pts.	Hermite
Symbol table	$k+1$	k	1	$B_n^{(2)}$ Partitions	Charlier
Stack	1	0	1	$C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan nb.	Tchebycheff

orthogonal

polynomials

(binomial type)

Scheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

Calcul du coût intégré

d'une structure de données

pour une séquence aléatoire

d'opérations primitives

Françon, Flajolet, Vuillemin (1980, ...)

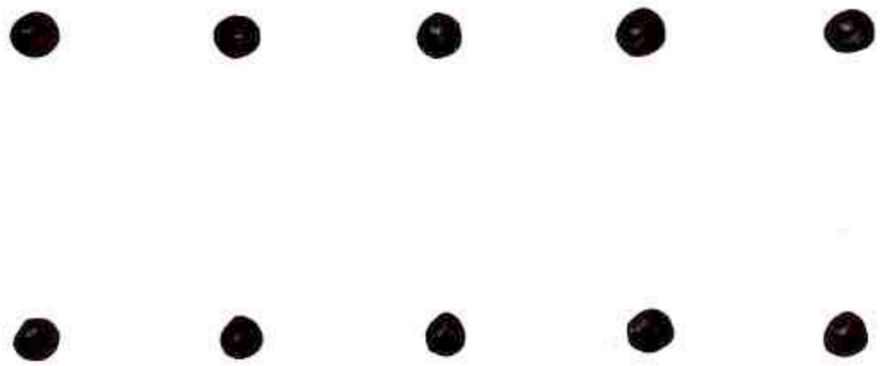
connaissant le coût moyen

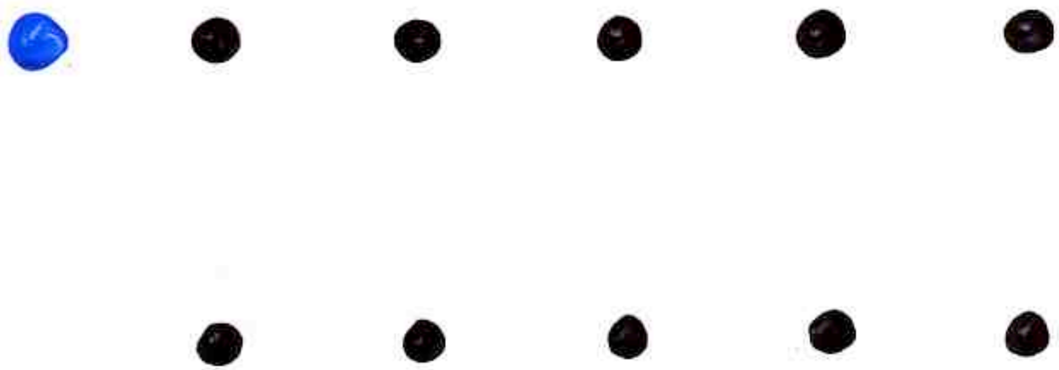
d'une opération primitive.

www.mathinfo06.iecn.u-nancy.fr

histories as:

combinatorial operators



A 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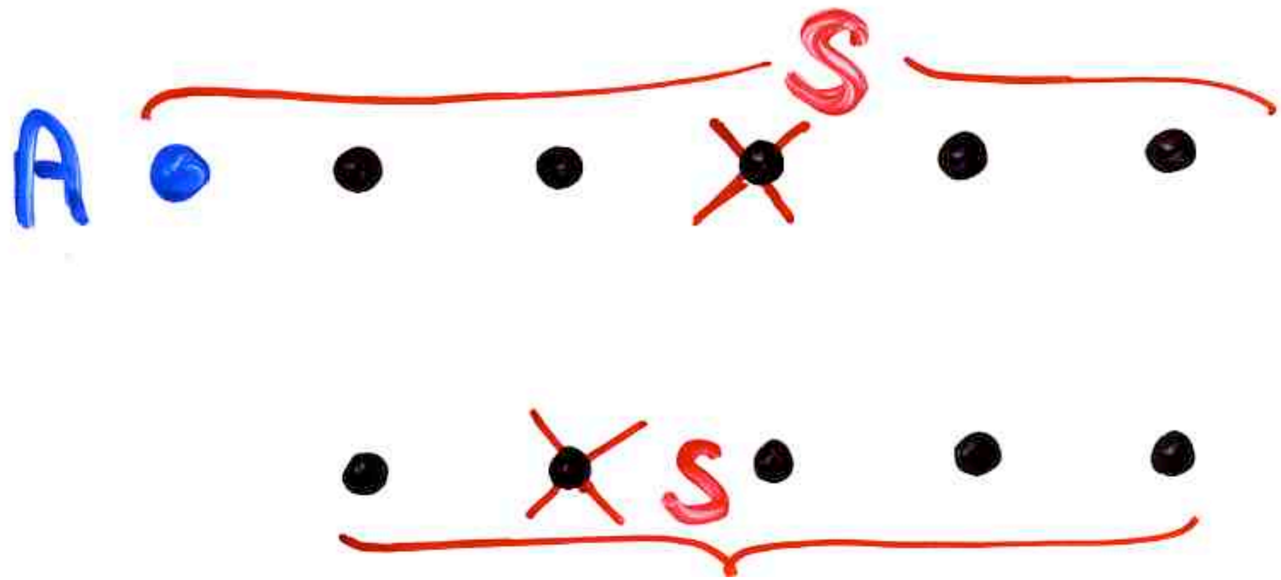
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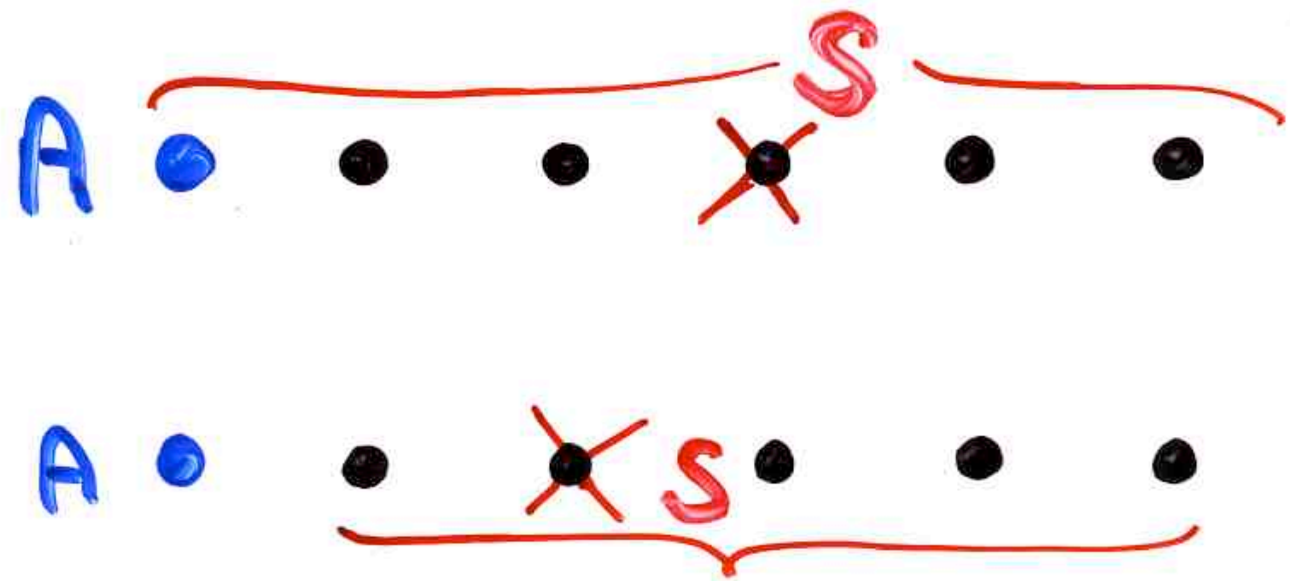
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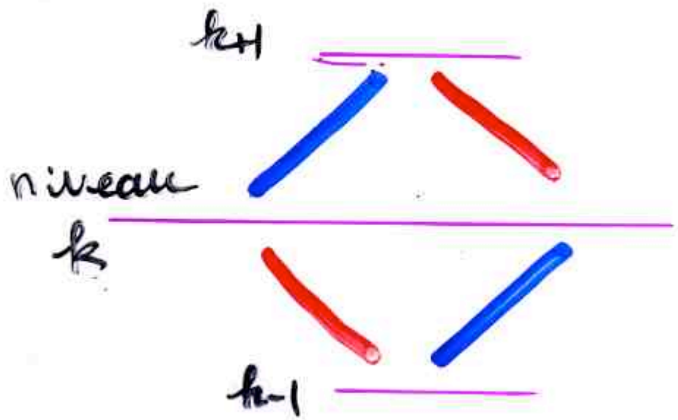


possibilités

AS $1 \times (k+1) = k+1$

SA $k \times 1 = k$

$$UD = DU + I$$

Weyl-Heisenberg algebra

$$UD = DU + Id$$

operators
creation - annihilation
particules

P. Flajolet, P. Blasiak
(2010)

normal ordering

$$UD = DU + I$$

$$w \in \{U, D\}^*$$

$$w = \sum_{i,j \geq 0} c_{i,j}(w)\, D^i U^j$$

K. Penson, I. Solomon
P. Blasiak, A. Horzela
G. Duchamp

$$\begin{cases} U \\ D \end{cases} \qquad \begin{cases} A \\ S \end{cases}$$

operators x x

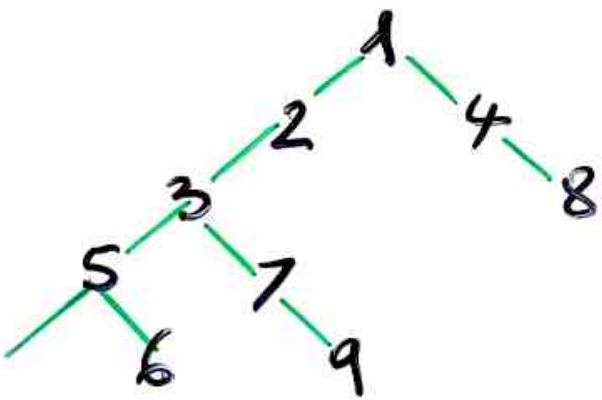
$\frac{d}{dx}(\ )$

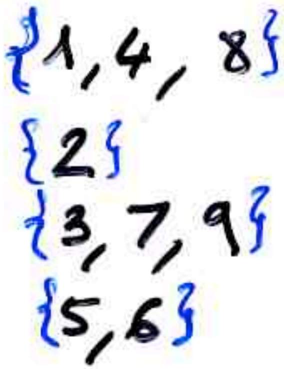
priority queue
Polya urn

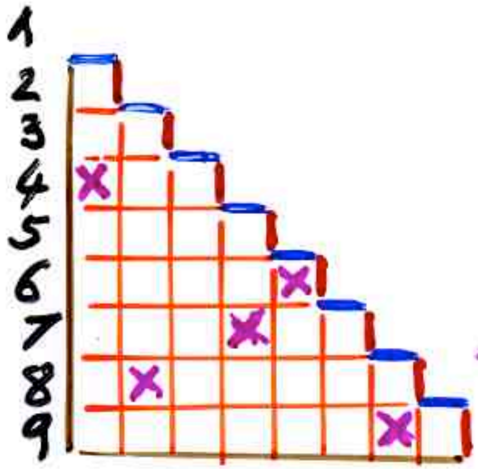
Prop. $w = (UD)^n$

$c_{k,k}(w) = S_{n+1,\,k+1}$ Stirling

nombre de partitions de $\{1, 2, \dots, n+1\}$ en $(k+1)$ blocs







histories as:

Polya urns

Polya urn model

y ○ → ● ●

x ● → ○ ○

history

$x \to yy \to yxx \to yyyx \to xxyyx \to xyyyyx$

nb of histories total $n!$

- starting ●, ending ○○○..○ — $\mathrm{sm}(-z)$
- starting ● ending ●●●....● $\mathrm{cm}(-z)$

elliptic functions ....

Apéry $\zeta(3) = \sum 1/n^3$

irrational

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\cdots}}}}$$

$$\varpi(n) = (2n+1)(17n(n+1)+5)$$

Séminaire Lotharingien de Combinatoire **54** (2006), Article B54g

THE FERMAT CUBIC, ELLIPTIC FUNCTIONS, CONTINUED FRACTIONS, AND A COMBINATORIAL EXCURSION

ERIC VAN FOSSEN CONRAD AND PHILIPPE FLAJOLET

Kindly dedicated to Gérard···Xavier Viennot on the occasion of his sixtieth birthday.

ABSTRACT. Elliptic functions considered by Dixon in the nineteenth century and related to Fermat's cubic, $x^3+y^3=1$, lead to a new set of continued fraction expansions with sextic numerators and cubic denominators. The functions and the fractions are pregnant with interesting combinatorics, including a special Pólya urn, a continuous-time branching process of the Yule type, as well as permutations satisfying various constraints that involve either parity of levels of elements or a repetitive pattern of order three. The combinatorial models are related to but different from models of elliptic functions earlier introduced by Viennot, Flajolet, Dumont, and Françon.

In 1978, Apéry announced an amazing discovery: "$\zeta(3) \equiv \sum 1/n^3$ *is irrational*". This represents a great piece of Eulerian mathematics of which van der Poorten has written a particularly vivid account in [59]. At the time of Apér
was known about the arithmetic nature of the zeta values at odd
unnaturally his theorem triggered interest in a whole range of pro
recognized to relate to much "deep" mathematics [38, 51]. Apéry's c
proof crucially depends on a continued fraction representation of $\zeta($

Lucelle (2005)
Séminaire Lotharingien
de Combinatoire
54th SLC

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\ddots}}}}, \tag{1}$$

where $\quad \varpi(n) = (2n+1)(17n(n+1)+5).$

$$\mathrm{sm}(z) = \mathrm{Inv} \int_0^z \frac{dt}{(1-t^2)^{2/3}}$$

$$\begin{cases} \mathrm{sm}' = \mathrm{cm}^2 \\ \mathrm{cm}' = -\mathrm{sm}^2 \end{cases} \qquad \begin{matrix} \mathrm{sm}(0) = 0 \\ \mathrm{cm}(0) = 1 \end{matrix}$$

$$\mathrm{sm}(z)^3 + \mathrm{cm}(z)^3 = 1$$

Dixon (1890)

Conrad (2002)

$$\int_0^\infty \mathrm{sm}(u)\, e^{-u/x}\, du = \cfrac{x^2}{1 + b_0 x^3 - \cfrac{1.2^2.3^2.4\, x^6}{1 + b_1 x^3 - \cfrac{4.5^2.6^2.7\, x^6}{1 + b_2 x^3 - \cdots}}}$$

$$b_n = 2(3n+1)\left((3n+1)^2 + 1\right)$$

Polya urn model

y ○ → ● ●

x ● → ○ ○

history

$\underline{x} \to y\underline{y} \to y\underline{x}x \to \underline{y}yyx \to x\underline{x}yyx \to xyyyyx$

nb of histories total $n!$

- starting ●, ending ○○○..○ — sm(−z)
- starting ● ending ●●●....● cm(−z)

$$sm(z) = z - 4\frac{z^4}{4!} - 160\frac{z^7}{7!} - 20800\frac{z^{10}}{10!} - \dots$$

$$cm(z) = z - 2\frac{z^3}{3!} - 40\frac{z^6}{6!} - 3680\frac{z^9}{9!} - \dots$$

nb of histories total $n!$

- starting ●, ending ○○○..○ $\quad -sm(-z)$
- starting ● ending ●●●....● $\quad cm(-z)$

- class of permutations
  based on parity

- 2- repeated permutations
  (with J. Françon) → sn, cn, dn
  (1989) Jacobian elliptic

  3 - repeated (*) permutations
  → $-\mathrm{sm}(-z)$ continued fraction

Jacobian elliptic functions

sn, cn, dn

X.V. (1980) Jacobi permutations

Dumont (1979) Flajolet alternating

Schett generation

cycle structure

- class of permutations

  based on parity

- 2- repeated permutations

  (with J. Françon) → sn, cn, dn

  (1989) Jacobian elliptic

  3 - repeated (*) permutations

  → $-sm(-z)$ continued fraction

P. F. with R. Bacher (2010)

pseudofactorial

$$a_{n+1} = (-1)^{n+1} \sum \binom{n}{k} a_k a_{n-k}$$

$$\sum_{n \geq 0} a_n z^n = \cfrac{1}{1 + 1z - \cfrac{3 \cdot 1^2 z^2}{1 - 1z + \cfrac{2^2 z^2}{1 + 3z + \cfrac{3 \cdot 3^2 z^2}{1 - 3z + \cfrac{4^2 z^2}{\cdots}}}}}$$

Weierstraß function $\wp$

lattice sum

addition formula
and
continued fraction

addition formula

Stieltjes - Rogers

$$\varphi(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$$

$$\varphi(x+y) = \sum_{k} \omega_k \varphi_k(x) \varphi_k(y)$$

$$\varphi_k(x) = \frac{x^k}{k!} + \varphi_{k,k+1} \frac{x^{k+1}}{(k+1)!} + \cdots$$

$$\sum_{n\geq 0} a_n z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{1 - b_2 z - \cdots}}}$$

$$\omega_k = \lambda_1 \lambda_2 \cdots \lambda_k$$

$$b_k = \varphi_{k,k+1} - \varphi_{k-1,k}$$

A Happy New Year 2010

Consider the **integer sequence** (p_n), which starts as

$$2, 144, 96768, 268240896, 2111592333312, 37975288540299264, \ldots$$

and is defined by sums over the **square lattice**,

$$p_n := (-1)^{n+1}(4n+3)! \left[\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right]^{-4n-4} \sum_{a,b=-\infty}^{+\infty} \left[(2a+1)+(2b+1)\sqrt{-1}\right]^{-4n-4}.$$

The following **continued fraction** expansion holds:

$$\sum_{n=0}^{\infty} p_n z^n = \cfrac{2}{1 - 2\cdot 2^2(2^2+5)\,z - \cfrac{2\cdot 3^2\cdot 4^2\cdot 5^2\cdot 6\,z^2}{1-2\cdot 6^2(6^2+5)\,z - \cfrac{6\cdot 7^2\cdot 8^2\cdot 9^2\cdot 10\,z^2}{1-2\cdot 10^2(10^2+5)\,z - \ddots}}}.$$

[A follow up to R. Bacher and P. Flajolet, *The Ramanujan Journal*, 2010, in press.]

Philippe Flajolet

extensions of
Flajolet theory
of continued fractions

Padé approximants

E. Roblet 1995

Approximants de Padé

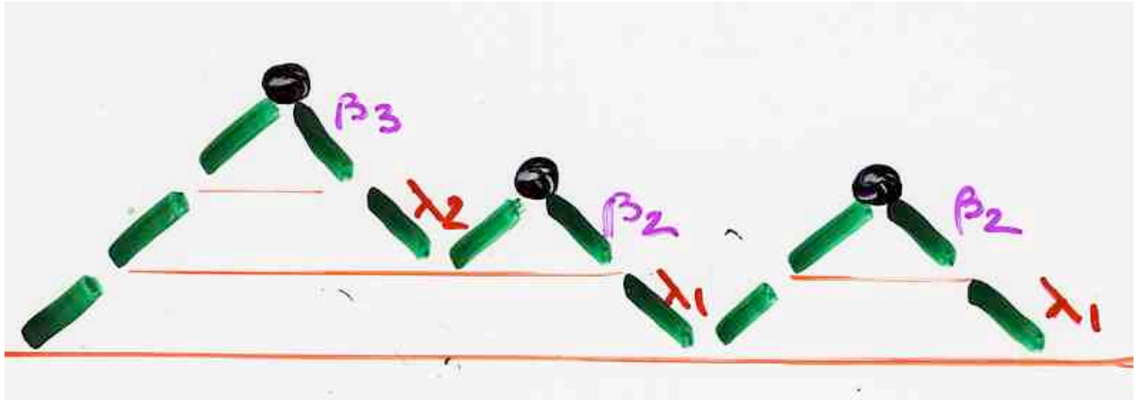
$$f(t) = a_0 + a_1 t + \quad + a_n t^n + \ldots$$

$$\simeq \frac{N_p(t)}{D_q(t)}$$

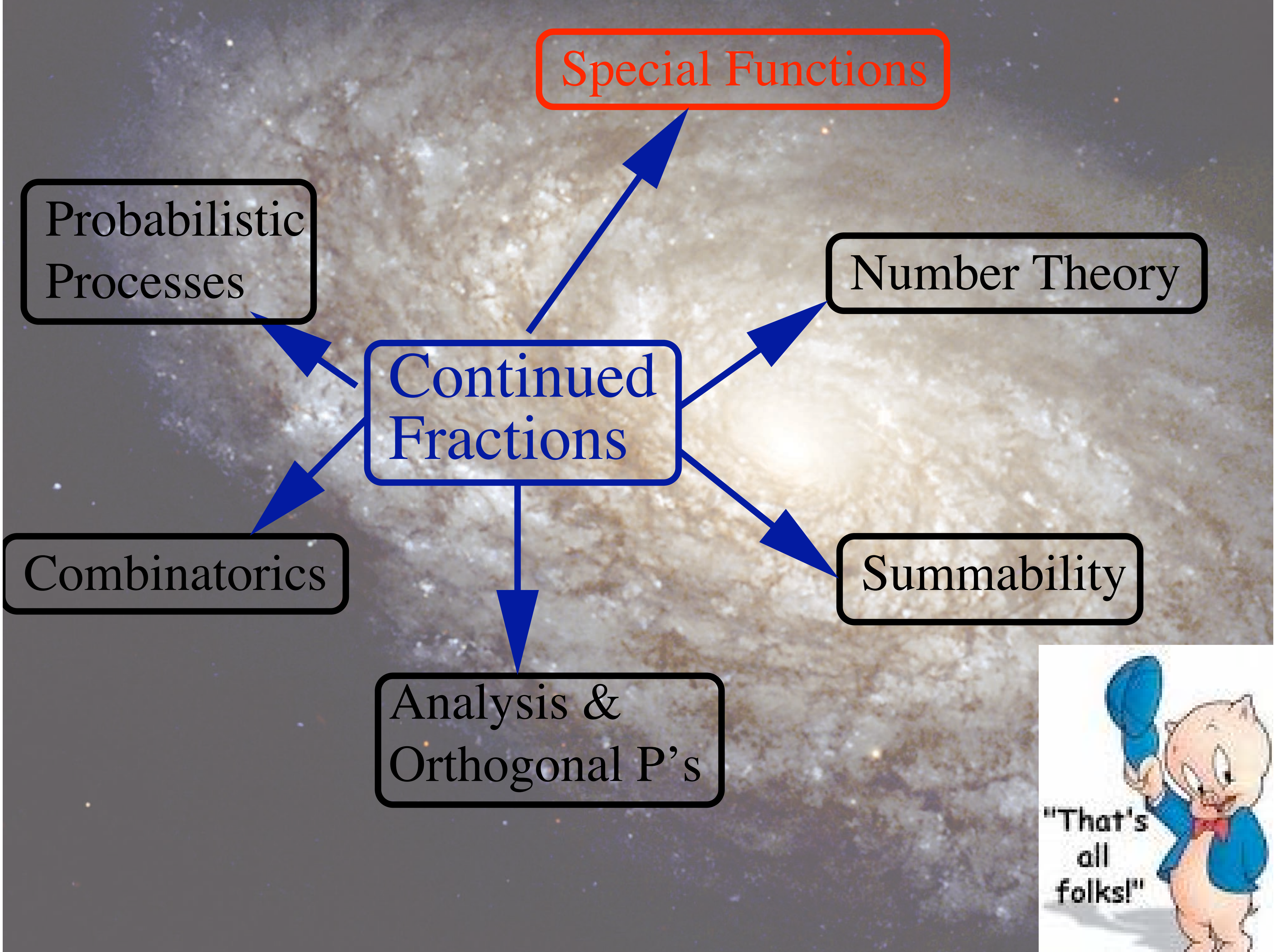
T- fractions

approximants in two points

Roblet, xgV. (1993)



$$\sum_{\substack{\omega \\ \text{Dyck}}} v(\omega)\, t^{|\omega|/2} = \cfrac{1}{1-(\beta_1-\lambda_1)t - \cfrac{\lambda_1 t}{1-(\beta_2-\lambda_2)t-\cfrac{\lambda_2 t}{\cdots}}}$$


Special Functions
Probabilistic Processes
Number Theory
Continued Fractions
Combinatorics
Summability
Analysis & Orthogonal P's
"That's all folks!"

some applications of
Flajolet theory
of continued fractions

physics

J. Bouttier, E. Guitter (2010)

planar maps and continued fractions

mobiles labeled trees

P. Di Francesco, R. Kedem

(2009)

Q-system

cluster algebra

paths

total positivity

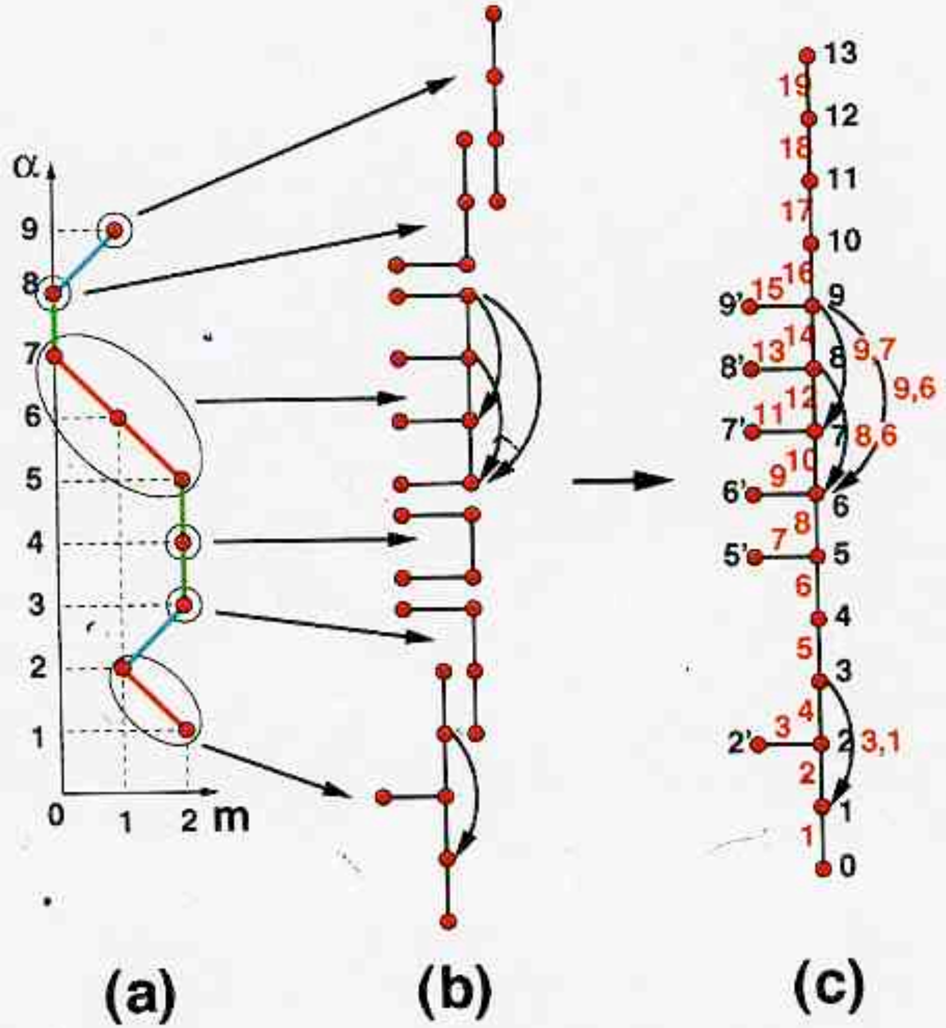
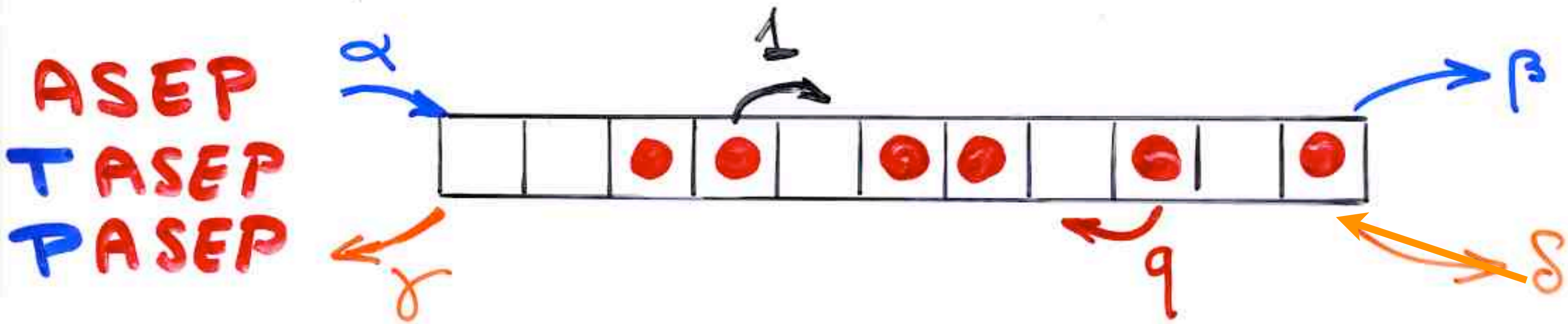


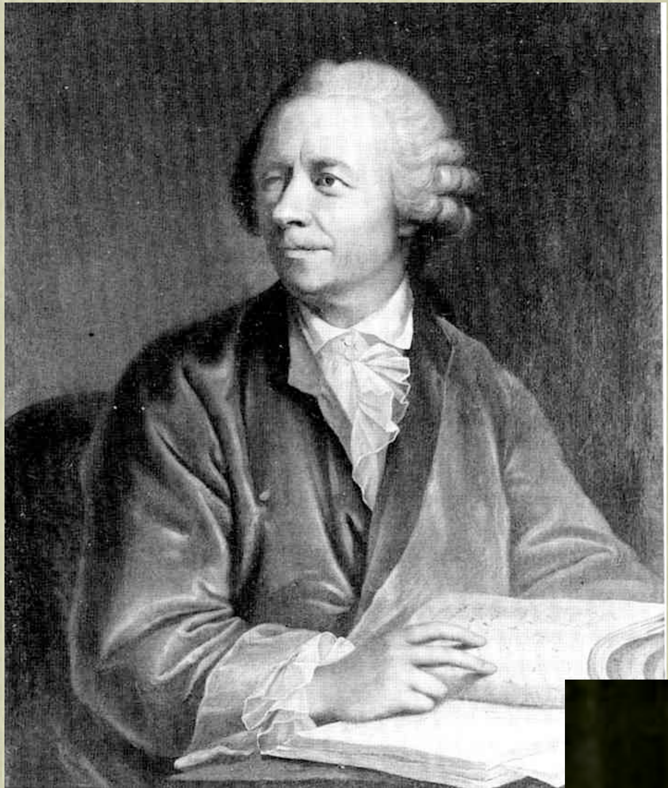
FIGURE 8. The Motzkin path $\mathbf{m} = (2,1,2,2,2,1,0,0,1)$ for $r = 9$ (a) is decomposed into $p = 6$ descending segments (12)(3)(4)(567)(8)(9) (circled, red edges). The corresponding graph pieces $\Gamma_{\mathbf{m}_i}$ are indicated in (b). They are to be glued "horizontally" for flat transitions (green edges) and "vertically" for ascending ones (blue edges). The resulting graph $\Gamma_{\mathbf{m}}$ is represented (c) with its vertex (black) and edge (red) labels.

PASEP

Partialy asymmetric exclusion process


ASEP
TASEP
PASEP
α
1
β
γ
q
δ


14:21
www.mathinfo06.iecn.u-nancy.fr



ॐ भूर्भुव: स्व:

तत्सवितुर्वरेण्यं ।

भर्गो देवस्य धीमहि,

धीयो यो न:

प्रचोद्यात् ॥

ॐ सरस्वत्यै नमः।

Merci Philippe !